Observability Reduction of Piecewise-affine Hybrid Systems

Mihály Petreczky and Jan H. van Schuppen

Abstract— **We present necessary conditions for observability of piecewise-affine hybrid systems. We also propose an observability reduction algorithm for transforming a piecewise-affine hybrid system to a hybrid system of possibly smaller dimension which satisfies the formulated necessary condition for observability.**

I. Introduction

In this paper we present necessary conditions for observability of *piecewise-affine systems (abbreviated as* ***PAHSs****)* and a dimensionality reduction procedure, based on this necessary condition.

A **PAHS** is a hybrid systems, continuous dynamics of which is determined by affine control systems, the reset maps are affine and the guards are polyhedral sets. The definition of **PAHS** adopted in this paper is essentialy the same as in [7], except that we allow the continuous state-space to be also a polyhedron as opposed to a polytope.

Contribution of the paper The contribution of the paper can be summarized as follows.

- **Necessary condition for observability** We formulate an algebraic necessary condition for observability of a general **PAHS**. This condition is a generalization of the well-known rank condition for linear systems.
- **Linear PAHSs** We introduce the class of *linear* **PAHSs**. A linear **PAHS** is a **PAHS** such that the control system in each discrete state is a linear (not affine) one, and lives on a polyhedron defined by linear inequalities, the reset maps are linear. For the class of linear **PAHSs** we propose necessary conditions, which are tighter than the ones for general **PAHS**. We refer to **PAHSs** which satisfy the latter condition as *weakly observable* ones. In particular, weak observability deals with discrete states with the same observable dynamics.
- **Observability reduction** We formulate a procedure for transforming an arbitrary **PAHS** to a linear, weakly observable **PAHS**. The dimension of the thus obtained **PAHS** is bounded by the dimension of the original **PAHS**. Hence, the proposed transformation can be viewed as model reduction procedure aimed at merging observationally equivalent states.

Approach The main idea is to represent a (linear) **PAHS** Σ as an output feedback interconnection of a *linear hybrid system without guards* (abbreviated as LHS, see [10], [9] for the definition) with an event generation device which detects crossing a guard, see Fig. 1. If we denote by H the LHS above, and by G the event generator, then Σ is observable, only if H is observable. Consider a LHS H_o such that H_o is observable and H_o realizes the same input-output behavior as H. If we consider the interconnection of H_o with the event generator G, we then obtain a **PAHS** Σ_o with the following property. The LHS component of Σ_o is observable and Σ_o realizes the same input-output behavior as Σ.

Observability of LHSs is well-understood [10], [9], and the computation of H_o can be carried out by an algorithm. Hence, the proposed necessary condition and observability reduction can be implemented numerically. In addition, the dimension of H_o is not greater than that of H. In fact, we believe that this link between **DPAHSs** and LHSsis interesting on its own right and will be useful for problems other than observability analysis of **PAHSs** .

Related work There is a vast literature on observability of various classes of piecewise-linear hybrid systems, without claiming completeness, see [1], [2], [3], [5], [13], [11] and many others. However, most of the existing literature deals with observability of hybrid systems which are related but not identical to **PAHSs**. For example, typically reset maps are not considered, and the switching mechanism is assumed to be arbitrary, rather than state induced. The thus obtained conditions are either not directly applicable to **PAHSs**, or yield sufficient conditions. We believe that the necessary conditions of the paper represent a new result with respect to the existing literature.

To the best of our knowledge, observability reduction of **PAHSs** was addressed only in [8]. The current paper is an extension of [8].

Outline Section II presents the terminology and notation used in the paper. Section III defines piecewise-affine hybrid systems and the related system theoretic concepts. Section IV presents the main results of the paper. Section V presents a sketch of the proof of the main results.

II. Preliminaries

Let Σ be a finite set, referred to as the *alphabet*. Σ^* denotes the set of finite *strings (words)* of elements of Σ, i.e. element of Σ^* is a sequence $w = a_1 a_2 \cdots a_k$, where $a_1, a_2, \ldots, a_k \in \Sigma$, and $k \geq 0$; k is *the length of* w and it is denoted by $|w|$. If $k = 0$, then w is the empty sequence (word), denoted by ϵ. The concatenation of the words $v = v_1 \cdots v_k$, and $w = w_1 \cdots w_m \in \Sigma^*$ is the word $vw = v_1 \cdots v_k w_1 \cdots w_m$. The empty word ϵ is a unit element for concatenation. i.e. $\epsilon w = w\epsilon = w$ for all $w \in \Sigma^*$.

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Denote by $\mathbb{N}$ the set of natural numbers including 0. Let T be the *real time-axis*, i.e. $T = [0, +\infty)$. Denote by $PC(T, \mathbb{R}^m)$ the set of piecewise-continuous maps (i.e. maps whose restriction to any finite interval is piecewise-continuous in the sense of [6]) with values in $\mathbb{R}^m$. For each $n > 0$ and $j = 1, 2, \ldots, n$, e_j is the jth standard unit basis vector of $\mathbb{R}^n$, i.e. $e_j = (\sigma_{1,j}, \sigma_{2,j}, \ldots, \sigma_{n,j})^T$, where $\sigma_{j,j} = 1$ and $\sigma_{i,j} = 0$ for $i \neq j$.

III. Piecewise-affine hybrid systems

Definition 1: A continuous-time piecewise-affine hybrid system (abbreviated as **PAHS**) is a hybrid system in the sense [12] of the following form

$$\Sigma \begin{cases} \dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t) + a_{q(t)} \\ y(t) = C_{q(t)}x(t) + c_{q(t)} \\ o(t) = \delta(q(t), \gamma(t)) \text{ and } q(t^+) = \delta(q(t), \gamma(t)) \\ x(t^+) = M_{q(t^+),\gamma(t),q(t)}x(t) + m_{q(t^+),\gamma(t),q(t)} \\ \gamma(t) = e \iff n_{q(t),e}^T x(t) = b_q, \text{ and } n_{q(t),e}^T \dot{x}(t) > 0 \\ h_0 = (q_0, x_0) \end{cases} \tag{1}$$

The various parameters are as follows

- $q(t) \in Q$ is the *discrete state* at time t, and Q is the *finite set of discrete states (modes)*,
- $o(t) \in O$ is the *discrete output* at time t, and O is the *finite set of discrete outputs*,
- $\gamma(t) \in \Gamma$ is *discrete event* at time t, and $\Gamma = \{1, 2, \ldots, \mathrm{E}\}$, $\mathrm{E} > 0$ *is the finite set of discrete events.*
- $\delta : Q \times \Gamma \to Q$ is the *discrete state-transition map*,
- $\lambda : Q \to O$ is the *discrete readout map*,
- For each $q \in Q$, the affine system is described by matrices $A_q \in \mathbb{R}^{n_q \times n_q}$, $B_q \in \mathbb{R}^{n_q \times m}$, $C_q \in \mathbb{R}^{p \times n_q}$
- The state $x(t)$ of Σ associated with the discrete state $q \in Q$ lives on the (convex) polyhedron $\mathcal{P}_q$ of the form
$$\mathcal{P}_q = \bigcap_{\gamma \in \Gamma} \{x \in \mathbb{R}^{n_q} \mid n_{q,\gamma}^T x \leq b_{q,\gamma}\}$$
where $n_{q,i} \in \mathbb{R}^{n_q}$, $b_{q,i} \in \mathbb{R}$. The facets of $\mathcal{P}_q$ are called *exit facets* of the polyhedron$\mathcal{P}_q$.
- $x(t) \in \mathbb{R}^{n_{q(t)}} = \mathcal{X}_{q(t)}$ is the *continuous state* at time t,
- $y(t) \in \mathbb{R}^p$, for $p > 0$, is the *continuous output* at time t, and $\mathbb{R}^p$ is the *space of continuous outputs*,
- $u(t) \in \mathbb{R}^m$, $m > 0$, is the *continuous input* at time t, and $\mathbb{R}^m$ is the *space of continuous inputs*,
- The transition between the different continuous state-spaces takes place via affine *reset map* $R_{q^+,\gamma,q}$, $q \in Q$, $\gamma \in \Gamma$, $q^+ = \delta(q, \gamma)$ where $R_{q^+,\gamma,q}(x) = M_{q^+,\gamma,q}x + m_{q^+,\gamma,q}$, $M_{q^+,\gamma,q} \in \mathbb{R}^{n_{q^+} \times n_q}$, and $m_{q^+,\gamma,q} \in \mathbb{R}^{n_{q^+}}$.
- $h_0 = (q_0, x_0)$, $x_0 \in \mathcal{P}_{q_0}$ is the initial state of Σ.

The *state space* $\mathcal{H}_\Sigma$ *of* Σ is $\mathcal{H}_\Sigma = \bigcup_{q \in Q} \{q\} \times \mathcal{P}_q$.

For the definition of evolution of a **PAHS** see [7]. Note that in contrast to [7], we also allow discrete outputs. In addition, we do not require the sets $\mathcal{P}_q$, $q \in Q$ to be polytopes, but only polyhedrons. However, the case of $\mathcal{P}_q$, $q \in Q$ being a polytope is a special case of the above definition. If Σ is a **PAHS** such that for each $q \in Q$, the polyhedron $\mathcal{P}_q$ is also a polytope, then we say that Σ is an ***PAHS*** *on polytopes*

The evolution of Σ takes place according to the definition [12], [7]. Assume that we feed in a $\mathbb{R}^m$-valued input signal $u(t) \in \mathbb{R}^m$. As long as the value of the discrete state q does not change, the continuous state and the continuous output change according to the affine system $\dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t) + a_{q(t)}$ and $y(t) = C_{q(t)}x(t) + c_{q(t)}$. The discrete state changes only when a discrete event occurs. A discrete event $e \in \Gamma$ takes place if the continuous state $x(t)$ is about to leave the polyhedron $\mathcal{P}_q$ through the exit facet associated with γ, i.e. $n_{q,\gamma}^T x(t^-) = b_{q,\gamma}$ and $n_{q,\gamma}^T \dot{x}(t^-) = n_{q,\gamma}^T (A_q x(t^-) + B_q u(t) + a_q) > 0$. Here $x(t^-)$ is the state just before the discrete event occurs, i.e. $x(t^-)$ is the left-hand side limit $x(t^-) = \lim_{s \to t^-} x(s)$. Then the new discrete state is determined by the discrete state-transition rule as $q^+ = \delta(q, \gamma)$. The new continuous state $x(t) = x(t^+) \in \mathbb{R}^{n_{q(t+)}}$ is obtained by applying the corresponding reset map, that is, $x(t^+) = M_{q^+,\gamma,q}x(t^-) + m_{q^+,\gamma,q}$. The discrete output is obtained from the discrete state by applying the discrete readout map, i.e. $o = \lambda(q)$. After that, the continuous state and output evolve according to the affine system associated with the new discrete state q^+.

Note that the event generation mechanism described above is non-deterministic, since the continuous state might cross several exit facets at the same time. In turn, this non-determinism could lead to non-uniqueness of state and output trajectory. In order to ensure existence of a unique state and output trajectory, we will parametrize **PAHSs** by so called *event generators*, i.e. maps which choose which of the potentially enabled events should be generated.

Definition 2 (Event-generator): An even generator is a partial map

$$G : \mathbb{R}^{\mathrm{E}} \times \mathbb{R}^{\mathrm{E}} \to \Gamma$$

such that for any $h = (z, \dot{z})$ the following holds.

1) If $G(h)$ is defined and $G(h) = e$, then $z_e = 0$ and $\dot{z}_e > 0$. Here z_e and $\dot{z}_e$ denote the eth entry of z and $\dot{z}$ respectively.
2) If $G(h)$ is not defined then for all $e \in \Gamma$, either $z_e \neq 0$ or $\dot{z}_e \leq 0$.

In order to use even generators to describe the behavior of a **PAHS** Σ, it is useful to introduce the following notation.

Definition 3 (Guard map): Assume that Σ is a **PAHS** of the form (1). The guard map of Σ is a map

$$G_\Sigma : \bigcup_{q \in Q} \{q\} \times \mathbb{R}^{n_q} \times \mathbb{R}^{n_q} \to \mathbb{R}^{\mathrm{E}} \times \mathbb{R}^{\mathrm{E}}$$

defined as follows. For any $(q, x, \dot{x})$,

$$G_\Sigma(q, x, \dot{x}) = \left(\begin{bmatrix} n_{q,1}^T x - b_{q,1} \\ n_{q,2}^T x - b_{q,2} \\ \vdots \\ n_{q,\mathrm{E}}^T x - b_{q,\mathrm{E}} \end{bmatrix}, \begin{bmatrix} n_{q,1}^T \dot{x} \\ n_{q,2}^T \dot{x} \\ \vdots \\ n_{q,\mathrm{E}}^T \dot{x} \end{bmatrix} \right)$$

The intuition behind the definition is as follows. G_Σ maps a hybrid state and the derivative of the continuous part to their scalar product with the normal vectors of the exit facet of the polyhedron. More precisely, if the current state is $h = (q,x)$, then $G_\Sigma(q,x,\dot{x})$ contains all the information the event generator G needs in order to decide which event to generate. That is, an event e is generated by Σ under the event generator G, if $G(G_\Sigma(q,x,\dot{x})) = e$ and no event is generated if $G(G_\Sigma(q,x,\dot{x}))$ is undefined.

Condition 1 of Definition 2 says that if G generates and event e, then this can happen only if the continuous state is about to cross one of the facets of the polyhedron. Condition 2 requires that G does not miss an event, i.e. if $G(h)$ is not defined, then the current continuous state is either inside the polyhedron or it is sliding along one of the facets.

Equipped with the notion of an event-generator, we can define a unique state trajectory of the **PAHS** Σ for each event generator. To this end, we introduce the following definition.

Definition 4: A *determinized* **PAHS** (abbreviated as **DPAHS**) is a pair (Σ, G), where Σ is a **PAHS** of the form (1) and G is an event generator.

Definition 5 (Time event sequence, [4]): A time event sequence is a strictly monotone sequence $(t_n)_{n=0}^{n^*}$ such that $n^* \in \mathbb{N} \cup \{+\infty\}$, $t_0 = 0$ and for all $0 < n < n^*$, $0 \le t_n < t_{n+1}$. If $n^* = +\infty$ then let $t_\infty = \sup\{t_n \mid n \in \mathbb{N}\}$.

Notation 1: For a set A, A^T is the set of all maps $f : [0, T_f) \to A$, where $T_f \in T \cup \{+\infty\}$.

Definition 6 (Input-to-state map): Consider a **DPAHS** (Σ, G), assume that Σ is of the form (1). Recall that $\mathcal{H}_\Sigma$ is the state-space of Σ. For any state $h = (q_{init}, x_{init})$, with $q_{init} \in Q$, $x_{init} \in \mathcal{P}_{q_{init}}$, define the *input-to-state* map of (Σ, G) induced by the state h as

$$x_{\Sigma,G,h} : PC(T, \mathbb{R}^m) \to \mathcal{H}_\Sigma^T$$

such that for any $u \in PC(T, \mathbb{R}^m)$, $x_{\Sigma,G,h}(u) : [0, T_{f,u}) \to \mathcal{H}_\Sigma$, where $T_{u,h}$ depends on u and h and the following holds. There exists a time event sequence $(t_n)_{n=0}^{n^*}$, such that

$$T_{u,h} = \begin{cases} +\infty & \text{if } n^* < +\infty \\ t_\infty & \text{if } n^* = +\infty \end{cases},$$

and there exists a (possibly empty) sequence $(\gamma_k)_{k=1}^{n^*}$ of events from Γ, such that the following holds.

1) $x_{\Sigma,G,h}(u)(0) = h$
2) For all $i \in \mathbb{N}$, $i \le n^*$, and all $t \in [0, T_{u,h})$ such that $t \in [t_i, t_{i+1})$, where for $i = n^* < +\infty$, $t_{n^*+1} = +\infty$,
$$x_{\Sigma,G,h}(u)(t) = (q(i), x_i(t))$$
where $q(i) \in Q$ and the map $x_i : [t_i, t_{i+1}) \to \mathcal{P}_{q(i)}$ satisfies the differential equation
$$\dot{x}_i(t) = A_{q(i)} x_i(t) + B_{q(i)} u(t) + a_{q(i)},$$
and for all $t \in [t_i, t_{i+1})$, $G(G_\Sigma(q(i), x(t), \dot{x}(t)))$ is undefined.
3) For all $i \in \mathbb{N}$, $i < n^*$, the following holds. If $t_{i+1} = 0$, then $(q(0), x(0^-) = h$, and if $t_{i+1} > 0$, then
$$\forall t \in [t_i, t_{i+1}) : x_{\Sigma,G,h}(u)(t) = (q(i), x_i(t))$$
$$x_i(t_{i+1}^-) = \lim_{t \to t_{i+1}^-} x_i(t)$$
Then,
$$G(G_\Sigma(q(i), x_i(t_{i+1}^-), \dot{x}_i(t_{i+1}^-))) = \gamma_{i+1} \in \Gamma$$
$$\dot{x}_i(t_{i+1}^-) = A_{q(i)} x_i(t_{i+1}^-) + B_{q(i)} u(t_{i+1}) + a_{q(i)}$$
$$x_{\Sigma,G,h}(u)(t_{i+1}) = (q(i+1), x_{i+1}(t_{i+1}))$$
$$q(i+1) = \delta(q(i), \gamma_{i+1})$$
$$x(t_{i+1}) = M_{q(i+1),\gamma_{i+1},q(i)} x_i(t_{i+1}^-) + m_{q(i+1),\gamma_{i+1},q(i)}$$

Remark 1 (Well-posedness of $x_{\Sigma,G,h}$): Note that $x_{\Sigma,G,h}$ need not be exist for all states h. Intuitively, $x_{\Sigma,G,h}$ exists, if no state trajectory starting from h allows generation of several consecutive events, such that one events occurs immediately after another one. Note that in practice for most of systems the latter scenario will not occur, and hence $x_{\Sigma,G,h}$ will exist. If each reset map $R_{q^+,\gamma,q}$ of Σ maps the boundary of the polyhedron $\mathcal{P}_q$ into the interior of the polyhedron $\mathcal{P}_{q^+}$, then it is easy to see that $x_{\Sigma,G,h}$ exists for any state h of Σ. Note that if $x_{\Sigma,G,h}$ exists, then it is unique.

Remark 2: In the definition above, $n^* = \infty$ and $t_\infty < +\infty$ corresponds to Zeno-behavior.

Assumption 1: In the sequel, for any **PAHS** Σ considered, it is assumed that for every event generator G, the input-to-state map x_{Σ,G,h_0}, induced by the initial state h_0 of Σ, exists.

Definition 7 (Input-output map): Assume that h is a state of the **DPAHS** (Σ, G), such that the map $x_{\Sigma,G,h}$ exists. Then the *input-output map* $y_{\Sigma,G,h}$ of (Σ, G) induced by h is a map

$$y_{\Sigma,G,h} : PC(T, \mathbb{R}^m) \to (O \times \mathbb{R}^p)^T$$

such that for all $u \in PC(T, \mathbb{R}^m)$ the domain of $y_{\Sigma,G,h}$ is the same as that of $x_{\Sigma,G,h}$, i.e. it is $[0, T_{u,h})$, and for every $t \in [0, T_{h,u})$, if $x_{\Sigma,G,h}(u)(t) = (q,x)$, then

$$y_{\Sigma,G,h}(u)(t) = (\lambda(q), C_q x)$$

Note that $y_{\Sigma,G,h}$ need not exist for all states h of Σ; the input-output map $y_{\Sigma,G,h}$ exists precisely when the input-to-state map $x_{\Sigma,G,h}$ exists. In particular, due to the Assumption 1, y_{Σ,G,h_0} exists, where h_0 is the initial state of Σ.

Definition 8 (Observability): Two states h_1 and h_2 of a **DPAHS** (Σ, G) are *indistinguishable*, if the input-outputs maps y_{Σ,G,h_1} and y_{Σ,G,h_2} exist and they are equal, i.e. if $y_{\Sigma,G,h_1} = y_{\Sigma,G,h_2}$. Note that the equality of $y_{\Sigma,G,h_1} = y_{\Sigma,G,h_2}$ also implies that the domains of $y_{\Sigma,G,h_1}(u)$ and $y_{\Sigma,G,h_2}(u)$ are the same for all $u \in PC(T, \mathbb{R}^m)$. The **DPAHS** (Σ, G) is called *observable*, if there exists no pair of distinct indistinguishable states, i.e. if for any $h_1, h_2 \in \mathcal{H}_\Sigma$ such that y_{Σ,G,h_i} and y_{Σ,G,h_2} exist, $y_{\Sigma,G,h_1} = y_{\Sigma,G,h_2}$ implies $h_1 = h_2$. The **PAHS** Σ is *observable*, if for any event generator G, the **DPAHS** (Σ, G) is observable.
Note that observability has implications only for those states, for which the input-output map exists.

Definition 9 (Realization): The *input-output map* of a **DPAHS** (Σ, G), denoted by $y_{\Sigma,G}$, is the input-output map of (Σ, G) induced by the initial state h_0 of Σ, i.e. $y_{\Sigma,G} = y_{\Sigma,G,h_0}$. An input-output map $f : PC(T, \mathbb{R}^m) \to (O \times \mathbb{R}^p)^T$ is said to be *realized* by (Σ, G), if $f = y_{\Sigma,G}$.
Recall that the dimension of a polyhedron $\mathcal{P}$ equals n, if there exists $n+1$ affinely independent elements of $\mathcal{P}$ which constitute an affine basis of the affine hull of $\mathcal{P}$.

Definition 10 (Dimension): For each discrete state $q \in Q$, denote by d_q the dimension of $\mathcal{P}_q$. Then the *dimension* of the **PAHS** Σ, denoted by $\dim \Sigma$, equals the pair $(|Q|, \sum_{q \in Q} d_q)$. In the sequel, we will use the following ordering on pairs of natural numbers;$(m,n) \le (p,q)$, if $m \le p$ and $n \le q$. That is, the pair (m,n) is *smaller than or equal to* the pair (p,q), if m is not greater than p and n is not greater than q. Notice that the above ordering is a *partial order*. That is, there can be two **PAHSs** with incomparable dimensions.

IV. Main results

Below we present the main results of the paper. Throughout this section, Σ *denotes a* ***PAHS*** *of the form (1).* In order to present the main results we need additional notation. For the intuitive understanding of this notation, it is useful to recall the approach of the paper. Namely, we can associate with Σ a linear hybrid system H_Σ without guards (abbreviated as LHS , [9], [10]). The state of H is the same as that of Σ. The output of H is then the output of Σ and the scalar product of the state with the normal vectors of the exit facets. The latter is simply the value of G_Σ at the current state. The formal definition of H_Σ will be presented in Section V. The motivation for defining H_Σ is that if Σ is a linear **PAHS** (the notion of linear **PAHS** will be defined below), then for any event generator G, (Σ, G) is a feedback interconnection of H_Σ and the event generator G, see Fig. 1.

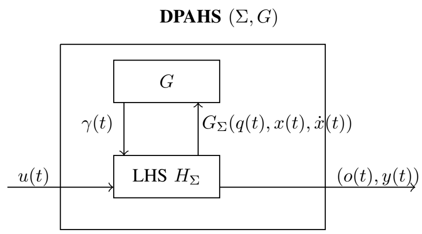

Fig. 1. **DPAHS** (Σ, G) as feedback interconnection of $H = H_\Sigma$ and G

Notation 2 (Augmented output matrices): Recall that $\Gamma = \{1, 2, \ldots, \mathrm{E}\}$. For any discrete state $q \in Q$,

$$\hat{C}_q = \begin{bmatrix} C_q^T & n_{q,1} & n_{q,2}, & \ldots & n_{q,\mathrm{E}} \end{bmatrix}^T \in \mathbb{R}^{p+\mathrm{E}} \qquad (2)$$

In other words, $\hat{C}_q$ is a block matrix, obtained by vertically 'stacking up' the matrix C_q and the normal vectors of the facets of $\mathcal{P}_q$. The matrix $\hat{C}_q$ corresponds to the readout matrix of the LHS H_Σ associated with Σ.

Notation 3 (Augmented event set): Let e be a symbol not in Γ, and define the set $\widetilde{\Gamma} = \Gamma \cup \{e\}$.

Notation 4 (Discrete state-transition map): We extend to discrete-state transition map δ to a map $\widetilde{\delta} : Q \times \widetilde{\Gamma}^* \to Q$ as follows. For any discrete state $q \in Q$ and sequence $w \in \widetilde{\Gamma}^*$ define the discrete state $\widetilde{\delta}(q,w)$ recursively as follows.

- If $w = \epsilon$, then $\widetilde{\delta}(q,w) = q$.
- If $w = v\sigma$ for some $\sigma \in \widetilde{\Gamma}$, $v \in \widetilde{\Gamma}^*$, then

$$\widetilde{\delta}(q, v\sigma) = \begin{cases} \widetilde{\delta}(q,v) & \text{if } \sigma = e \\ \delta(\widetilde{\delta}(q,v), \gamma) & \text{if } \sigma = \gamma \in \Gamma \end{cases}$$

By abuse of notation, we denote the extension $\widetilde{\delta}$ of the discrete state-transition map by δ as well.

Notation 5 (Product of matrices): For any $q \in Q$ and sequence $w \in \widetilde{\Gamma}^*$ define the $n_{\hat{q}} \times n_q$ matrix $\Pi(q,w)$, where $\hat{q} = \delta(q,w)$, recursively as follows.

- If $w = \epsilon$, then $\Pi(q,w) = I_{n_q}$, where I_{n_q} is the $n_q \times n_q$ identity matrix.
- If $w = v\sigma$ for some $\sigma \in \widetilde{\Gamma}$, $v \in \widetilde{\Gamma}^*$, then

$$\Pi(q, v\sigma) = \begin{cases} A_q \Pi(q,v) & \text{if } \sigma = e \\ M_{\delta(q,\gamma),\gamma,q} \Pi(q,v) & \text{if } \sigma = \gamma \in \Gamma \end{cases}$$

Define the $p\mathrm{E} \times n_q$ output matrix $\mathcal{O}(q,w)$ as follows

$$\mathcal{O}(q,w) = \hat{C}_{\delta(q,w)} \Pi(q,w)$$

Next, we introduce the generalization of the notion of Markov parameters for **PAHSs**.

Definition 11 (Markov parameters): The *Markov parameter* of Σ indexed by discrete state $q \in Q$, sequences $w \in \widetilde{\Gamma}^*$ and $v \in \Gamma^*$ is defined as the following matrix

$$\mathbb{M}_q(w,v) = \mathcal{O}(q,w) B_{\delta(q,v)} \in \mathbb{R}^{p\mathrm{E} \times m}$$

Note that in the definition of the markov parameter $\mathbb{M}_q(w,v)$, the sequence v is composed only of discrete events, while w can contain the addition symbol e. The above definition is inspired by theory of LHSs, in fact the Markov-parameters of Σ are the Markov-parameters of the LHS H_Σ. Intuitively, the Markov parameter $\mathbb{M}_q(w,v)$ corresponds to a certain derivative of the continous output generated from the discrete state $\delta(q,v)$, with respect to the event times.

Finally, for any discrete state $q \in Q$ of Σ we define the generalization of observability subspace.

Definition 12 (Observability kernel): For any discrete state $q \in Q$ of Σ define the observability subspace $O_{\Sigma,q}$ of Σ as a subset of $\mathbb{R}^{n_q}$ of the following form

$$O_{\Sigma,q} = \bigcap_{w \in \widetilde{\Gamma}^*} \ker \mathcal{O}(q,w)$$

The space $O_{\Sigma,q}$ is a generalization of the observability space for linear systems. In fact, $O_{\Sigma,q}$ is contained in the observability space of the linear system (A_q, C_q). However, $O_{\Sigma,q}$ also takes into account the output after first, second, etc. discrete-state transition. That is why products of the matrices of the affine subsystems and of the reset maps are considered too. The space $O_{\Sigma,q}$ is identical to the observability space $O_{H_\Sigma,q}$ ([10], [9]) of the LHS H_Σ associated with Σ.

Definition 13 (Full-dimensional ***PAHS****):* A **PAHS** Σ of the form (1) is called *full-dimensional*, if for any $q \in Q$, the dimension of the polyhedron $\mathcal{P}_q$ equals n_q.

Definition 14 (Complete ***PAHS****):* We say that a **PAHS** Σ is *complete*, if for any state h of Σ, and for any event generator G, the input-output map $y_{\Sigma,G,h}$ exists.

With the notation above, we are ready to present the first one of the main theorem.

Theorem 1: Assume Σ is a full-dimensional and complete **PAHS**. If Σ is observable, then for each $q \in Q$, $O_{\Sigma,q} = \{0\}$.

Theorem 1 can be viewed as a direct extension of the results of [8]. Below we present a more tight necessary condition for observability.

Definition 15 (Linear ***PAHSs****):* A **PAHS** of the form (1) is called *linear*, if the for all $q \in Q$,

- $a_q = 0$, $c_q = 0$ and for all $\gamma \in \Gamma$, $m_{\delta(q,\gamma),\gamma,q} = 0$. That is, the vector field for $u = 0$, the readout map and the reset maps associated with q are linear (not affine).
- For each exit facet indexed by $\gamma \in \Gamma$, $b_{q,\gamma} = 0$.

In order to simplify the presentation, we introduce the following definition.

Definition 16 (Weak observability): The **PAHS** Σ is called *weakly observable*, if the following conditions hold.

(i) For each two states $s_1, s_2 \in Q$, $s_1 = s_2$ if and only if

$$\begin{aligned} &\forall v \in \Gamma^* : \lambda(\delta(s_1, v)) = \lambda(\delta(s_2, v)), \text{ and} \\ &\forall v \in \Gamma^*, \forall w \in \widetilde{\Gamma}^* : \mathbb{M}_{s_1}(w, v) = \mathbb{M}_{s_2}(w, v) \end{aligned} \tag{3}$$

(ii) For each $q \in Q$, the zero vector is the only element of the subspace $O_{\Sigma,q}$, i.e. $O_{\Sigma,q} = \{0\}$.

Later on we will show that weak observability is equivalent to observability of the LHS H_Σ associated with Σ.

Theorem 2: Assume Σ is a full-dimensional, linear and complete **PAHS**. If Σ is observable, then it is weakly observable.

Remark 3: It is possible to extend Theorem 1–2 to hold for **PAHSs** which are not complete. To this end, one has to define input-output maps for the case when several consecutive events may occur with no time lag between them. In this paper we restrict attention to complete **PAHSs** only in order to avoid complicated notation.

The intuition behind Theorem 2 is the following. Since (Σ, G) is a feedback interconnection of H_Σ with G, if (Σ, G) is observable, then so is H_Σ. In turn, weak observability of Σ is equivalent to observability of H_Σ.

Remark 4: Weak observability of Σ can be checked numerically, since weak observability of Σ is equivalent to observability of the LHS associated with Σ, and by [9], the latter can be checked numerically.

Theorem 3 (Observability reduction): Any (hence not necessarily linear) **PAHS** Σ of the form (1) can be transformed to a weakly observable, linear, and full dimensional **PAHS** Σ_o, such that

(a) For any discrete-event generator G (Σ_o, G) realizes the same input-output map as (Σ, G), i.e. $y_{\Sigma,G} = y_{\Sigma_o,G}$,

(b) $\dim \Sigma_o \leq (|Q|, \sum_{q \in Q}(n_q + 1))$. If Σ is a *linear full-dimensional **PAHS***, then $\dim \Sigma_o \leq \dim \Sigma$.

Moreover, Σ_o can effectively be computed from Σ.

Note that the **PAHS** Σ_o above need not be complete. The intuition behind the theorem is as follows. It is always possible to convert a **PAHS** to a linear full-dimensional **PAHS** realizing the same input-output map, see Section V. Hence, without loss of generality we can assume that Σ is linear and full dimensional. It is possible to convert the LHS H_Σ of Σ to an observable LHS H_o which realizes the same input-output behavior. If in the feedback loop we replace H_Σ with H_o, then we obtain a **PAHS** Σ_o (see Fig. 2), which has the same input-output behavior as Σ, but the associated LHS of which is observable, i.e. Σ_o which is weakly observable.

V. Proof of the main result

In this section we present the proof of the main result. In §V-A we recall from [10], [9] the notion of LHSs. In §V-B we present the relationship between **PAHSs** and LHSs. In §V-C we present the behavior preserving transformation of a **PAHS** to a linear full dimensional one. Finally, in §V-D we sketch the proofs of the main results.

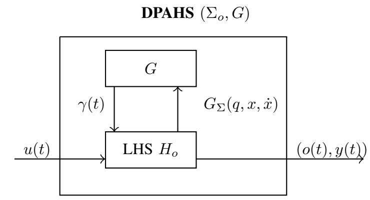

Fig. 2. Observability reduction: (Σ_o, G) and (Σ, G) from Fig. 1 are input-output equivalent.

A. Linear hybrid systems

The current section is a review of [9], [10].

Definition 17 (Linear hybrid systems,[9], [10]): A linear hybrid system (abbreviated as LHS) is a hybrid system *without guards* of the form

$$H : \begin{cases} \dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t), \quad y(t) = \hat{C}_{q(t)}x(t) \\ q(t+) = \delta(q(t), \gamma(t)), \ \ o(t) = \lambda(q(t)) \\ x(t+) = M_{q(t+),\gamma(t),q(t)}x(t) \\ h_0 = (q_0, x_0) \end{cases} \tag{4}$$

- $q(t) \in Q$ is the *discrete state* at time t, and Q is the *finite set of discrete states (modes)*,
- $o(t) \in O$ is the *discrete output* at time t, and O is the *finite set of discrete outputs*,
- $\gamma(t) \in \Gamma$ is *discrete event* at time t, and Γ *is the finite set of discrete events*,
- $\delta : Q \times \Gamma \to Q$ is the *discrete state-transition map*,
- $\lambda : Q \to O$ is the *discrete readout map*,
- $A_q \in \mathbb{R}^{n_q \times n_q}$, $B_q \in \mathbb{R}^{n_q \times m}$, $\hat{C}_q \in \mathbb{R}^{\hat{p} \times n_q}$ are the matrices, and $\mathcal{X}_q = \mathbb{R}^{n_q}$, $n_q > 0$, is *the continuous state-space*, of the linear system in $q \in Q$,
- $x(t) \in \mathbb{R}^{n_{q(t)}} = \mathcal{X}_{q(t)}$ is the *continuous state* at time t,
- $y(t) \in \mathbb{R}^{\hat{p}}$, for $\hat{p} > 0$, is the *continuous output* at time t, and $\mathbb{R}^{\hat{p}}$ is the *space of continuous outputs*,
- $u(t) \in \mathbb{R}^m$, $m > 0$, is the *continuous input* at time t, and $\mathbb{R}^m$ is the *space of continuous inputs*,
- the matrices $M_{\delta(q,\gamma),\gamma,q} \in \mathbb{R}^{n_{\delta(q,\gamma)} \times n_q}$, $q \in Q$, $\gamma \in \Gamma$, specify the linear *reset maps*.
- $h_0 = (q_0, x_0)$ – initial state.

The *state space* $\mathcal{H}_H$ *of* H is $\mathcal{H}_H = \bigcup_{q \in Q}\{q\} \times \mathcal{X}_q$.

In the rest of this section, H denotes a LHS of the form (4). The state and output of an LHS evolves as follows. If no discrete event occurs, the evolution is governed by the linear system of the current discrete state. As soon as a discrete event arrives, a discrete-state transition occurs, the continuous state is reset according to the reset map, and the system resumes its evolution according to the linear system of the new discrete state. *Note that for LHSs discrete events are external inputs, and there are no guards* For the formal description, we need the following notion.

Definition 18 (Timed sequences): A *timed sequence of events* is a sequence

$$w = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k) \quad (5)$$

where $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma$, $k \geq 0$, and $t_1, t_2, \ldots, t_k \in T$. We denote the set of all such sequences by $(\Gamma \times T)^*$. If $k = 0$, then w is the *empty sequence*, and it is denoted by ϵ.
The interpretation of w above is the following. The event γ_i took place *after* the event γ_{i-1} and t_i is the *elapsed time between the arrival of* γ_{i-1} *and the arrival of* γ_i. If $i = 1$, then t_1 is the arrival time of the first event γ_1.

Notation 6: Denote the set of inputs of an LHS by $\mathbf{U} = PC(T, \mathbb{R}^m) \times (\Gamma \times T)^* \times T$.

Definition 19 (State evolution): Consider a triple $\underline{u} = (u, w, t_{k+1}) \in \mathbf{U}$, where w is of the form (5). For a state $h = (q, x) \in \mathcal{H}_H$ of H, define the *state* $\xi_H(h, u, w, t_{k+1})$ *reached from* h *with inputs* (u, w, t_{k+1}) *at time* $\sum_{j=1}^{k+1} t_j$ recursively on k as follows.

For $k = 0$, let q and $x(t) \in \mathcal{X}_q$ be the solution of (6)

$$\dot{x}(t) = A_q x(t) + B_q u(t) \quad (6)$$

with $x(0) = x$, and set $\xi_H(h, u, \epsilon, t_1) = (q, x(t_1))$.

If for $v = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_{k-1}, t_{k-1}) \in (\Gamma \times T)^*$, $k > 0$, the state $\xi_H(h, u, v, t_k) = (q_k, x_k)$ is already defined, then set $q_k = \delta(q_{k-1}, \gamma_k)$, and let $z(t) \in \mathcal{X}_{q_k}$ be the solution of (7) with the initial condition $z(0) = M_{q_k, \gamma_k, q_{k-1}} x_k$.

$$\dot{z}(t) = A_{q_k} z(t) + B_{q_k} u(t + \sum_{1}^{k} t_j) \quad (7)$$

Set then $\xi_H(h, u, w, t_{k+1}) = (q_k, z(t_{k+1}))$.
Note that in $\xi_H(h, u, w, t_{k+1})$, the argument $t_{k+1} \in T$ denotes the time which has passed since the arrival of the last event γ_k. Next, we will define the input-output behavior of LHSs induced by a state.

Definition 20 (Input-output maps): The *input-output map* $\upsilon_{H,h}$ *and the continuous input-output map* $y_{H,h}$ *of* H *induced by the state* $h \in \mathcal{H}_H$ are maps

$$\upsilon_{H,h} : \mathbf{U} \rightarrow O \times \mathbb{R}^{\hat{p}} \text{ and } y_{H,h} : \mathbf{U} \rightarrow \mathbb{R}^{\hat{p}}$$

such that for each $\underline{u} \in \mathbf{U}$, if $(q, x) = \xi_H(h, \underline{u})$, then

$$\upsilon_{H,h}(\underline{u}) = (\lambda(q), \hat{C}_q x) \text{ and } y_{H,h}(\underline{u}) = \hat{C}_q x$$

Definition 21 (Realization): The LHS H is is a realization of the map $f : \mathbf{U} \rightarrow O \times \mathbb{R}^{\hat{p}}$ if f equals the input-output map of H induced by the initial state, i.e. $f = \upsilon_{H,h_0}$.

Definition 22 (Observability): Two distinct states $h_1 \neq h_2 \in \mathcal{H}_H$ of the LHS H are *indistinguishable*, if the input-output maps induced by h_1 and h_2 are equal, i.e. $\upsilon_{H,h_1} = \upsilon_{H,h_2}$. The system is called *observable*, if it has no pair of distinct indistinguishable states.

B. *Relationship between* ***PAHSs*** *and LHSs*

We start by defining the LHS associated with a **PAHS**.

Definition 23 (LHS associated with ***PAHS*** *):* Consider a linear **PAHS** Σ of the form (1). Define the LHS H_Σ associated with Σ as the LHS of the form (4) such that the following holds.

- The set of discrete states, outputs, events, the discrete state-transition map and the discrete readout map of H_Σ are all the same as those of Σ,
- The matrices A_q, B_q and $M_{\delta(q,\gamma),\gamma,q}$, $q \in Q, \gamma \in \Gamma$, of H_Σ are the same as those of Σ.
- $\hat{p} = p|\Gamma|$, and the readout matrix $\hat{C}_a$ of H_Σ is the augmented readout map of Σ, as defined in (2).
- The initial state of H_Σ is the same as that of Σ.

Conversely, we can associate a **PAHS** with each LHS.

*Definition 24 (****PAHS*** *associated with LHS):* Consider an LHS H of the form (4), such that $\hat{p} = p + |\Gamma|$. The **PAHS** Σ_H associated with H is a **PAHS** of the form (1), such that the following holds.

- The set of discrete states Q, discrete outputs O, events Γ, the state-transition map δ, and the readout map λ is the same for Σ_H as for H.
- For all $q \in Q$, the matrices $A_q, B_q, M_{\delta(q,\gamma),\gamma,q}$ of $\Sigma_{H,}$ are the same as those of H. The vectors a_q, c_q and $m_{\delta(q,\gamma),\gamma,q}$ are all zero.
- For all $q \in Q$, the matrix C_q of Σ_H is formed by the first p rows of $\hat{C}_q$, i.e. $\hat{C}_q = \begin{bmatrix} C_q^T & n_{1,q} & \ldots & n_{\mathrm{E},q} \end{bmatrix}^T$.
- For each $q \in Q$, the polyhedron $\mathcal{P}_q$ of Σ_H is

$$\mathcal{P}_q = \bigcap_{\gamma \in \Gamma} \{x \in \mathbb{R}^{n_q} \mid n_{q,\gamma}^T x \leq 0\}$$

  where $n_{q,i}^T$ is the $p + i$th row of $\hat{C}_q$.
- The initial state of Σ_H is the same as that of H.

Intuitively, Σ_H is obtained from H by defining the polyhedron for each discrete state $q \in Q$ as the polyhedron, normal vectors of the exit facets of which correspond to the last $|\Gamma|$ rows of the readout matrix $\hat{C}_q$ of H. Notice that the correspondence of Definition 24 is dual to the one of Definition 23.

Lemma 1: With the notation of Definition 23, the LHS H_{Σ_H} associated with Σ_H equals H.
Next, we present a result which relates the state and output of **PAHS** with the state and output of the associated LHS. To this end, we need the following. Consider a **DPAHS** (Σ, G) such Σ is linear and it is of the form (1). For an input $u \in PC(T, \mathbb{R}^m)$, state h of Σ, such that $x_{\Sigma,G,h}$ exists, consider the domain $T_{u,h}$ of $x_{\Sigma,G,h}$. For any $t \in [0, T_{u,h})$, define the pair $\mathbf{EV}_{\Sigma,G}(h, u, t) = (s, \hat{t}) \in (\Gamma \times T)^* \times T$ such that s is the timed event sequence generated by (Σ, G) on the interval $[0, t]$, if started in state h and fed input u, and no event occurs on $(t - \hat{t}, t]$. Formally, if $n^* = 0$, then $\hat{t} = t$ and $s = \epsilon$; if $n^* > 0$, and $t \in [t_i, t_{i+1})$ for some $i = 0, 1, \ldots, n^*$ then $s = (\gamma_1, t_1) \cdots (\gamma_k, t_k - t_{k-1})$ and $\hat{t} = t - t_k$. Note that for $t \in [0, T_{u,h})$, $\mathbf{EV}_{\Sigma,G}(h, u, t)$ depends only on $\upsilon_{H,h}$, and if $\upsilon_{H,h}(u, \mathbf{EV}_{\Sigma,G}(h, u, t)) = (o, (y, z))$ with $o \in O, y \in \mathbb{R}^p, z \in \mathbb{R}^{\mathrm{E}}$, then $y_{\Sigma,G,h}(u)(t) = (o, y)$. Combining these remarks, we get the following.

Lemma 2: Assume that Σ_i, $i = 1, 2$ are linear **PAHSs** and let H_{Σ_i} be the LHSs associated with Σ_i, $i = 1, 2$. Assume that h_i is a state of Σ_i and that for any event generator G, y_{Σ_i,G,h_i} exists, for $i = 1, 2$, If $\upsilon_{H_{\Sigma_1},h_1} = \upsilon_{H_{\Sigma_2},h_2}$, then for any event generator G, $y_{\Sigma_1,G,h_1} = y_{\Sigma_2,G,h_2}$.

Next, we state a result, which is interesting on its own right.

Theorem 4: Assume that Σ is a linear, full-dimensional and complete **PAHS** and let G be any event generator. If Σ is observable, then the associated LHS H_Σ is observable.

For the proof of Theorem 4, we need the following.

Lemma 3: Assume that H is an LHS of the form (4). For any state (q_i, x_i) of H, $i = 1, 2$, if $\upsilon_{H,(q_1,x_1)} = \upsilon_{H,(q_2,x_2)}$, then $\upsilon_{H,(q_1,0)} = \upsilon_{H,(q_2,0)}$. In addition, $\upsilon_{H,(q,x_1)} = \upsilon_{H,(q,x_2)}$ is equivalent to $y_{H,(q,x_1-x_2)}(0, w, t) = 0$ for all $w \in (\Gamma \times T)^*$, $t \in T$. Moreover, $y_{H,(q,x)}(0, w, t)$ is linear in x.

The proof of Lemma 3 follows from the proof of Theorem 2, [10]. In addition, we need the following algebraic result.

Lemma 4: If $\mathcal{P}$ is a full dimensional polyhedron on $\mathbb{R}^n$ and W is a proper (non-zero) linear subspace $\mathbb{R}^n$, then there exist $x_1, x_2 \in \mathcal{P}$ such that $x_1 - x_2 \in W$.

Proof: [Sketch of proof of Theorem 4] Assume that Σ is observable, but $H = H_\Sigma$ is not observable. The latter means that there exists two states $h_i = (q_i, x_i)$, $i = 1, 2$, such that $\upsilon_{H,h_1} = \upsilon_{H,h_2}$. By Lemma 3, it then implies that $\upsilon_{H,(q_1,0)} = \upsilon_{H,(q_2,0)}$. By linearity of Σ, $0 \in \mathcal{P}_{q_i}$ for $i = 1, 2$. But then from Lemma 2 we obtain that $y_{\Sigma,G,(q_1,0)} = y_{\Sigma,G,(q_2,0)}$. Hence, if $q_1 \neq q_2$, we obtain that Σ is not observable, which is a contradiction.

Assume that $q_1 = q_2$. Then by Lemma 3, $\upsilon_{H,h_1} = \upsilon_{H,h_2}$ is equivalent to $y_{H,(q,x_1-x_2)} = 0$. Denote by W_q the set of all elements $x \in \mathbb{R}^{n_q}$, such that $y_{H,(q,x)} = 0$. From Lemma 3 it follows that W_q is a linear space and $x_1 - x_2 \in W_q$, $x_1 \neq x_2$ implies that W_q is not trivial. Then by Lemma 4 we get that there exist $x_1, x_2 \in \mathcal{P}_q$ such that $x_1 - x_2 \in W_q$. But it implies that $y_{H,(q,x_1-x_2)} = 0$ and hence $\upsilon_{H,(q,x_1)} = \upsilon_{H,(q,x_2)}$. But the latter, together with the fact that (q, x_1) and (q, x_2) are both states of Σ and Lemma 2 implies that $y_{\Sigma,G,(q,x_1)} = y_{\Sigma,G,(q,x_2)}$. But this contradicts to observability of Σ. ■

C. Conversion of a PAHS to a linear full-dimensional one

In this section, Σ *is a* ***PAHS*** *of the form* (1). First, we present the transformation of Σ to a full dimensional **PAHS**.

Definition 25: Define the *full-dimensional* **PAHS** $F(\Sigma)$ associated with Σ as follows.

- The discrete state and output sets, the discrete state-transition and readout maps of Σ and $F(\Sigma)$ are identical.
- For each $q \in Q$, let d_q be the dimension of the affine span of elements of $\mathcal{P}_q$. Then the continuous state space of $F(\Sigma)$ in q is $\hat{\mathcal{P}}_q$ and the affine system is
$$\dot{\hat{x}}(t) = \hat{A}_q\hat{x}(t) + \hat{B}_q u(t) + \hat{a}_q$$
$$y(t) = \hat{C}_q\hat{x}(t) + \hat{c}_q$$
The reset map $\hat{R}_{q^+,\gamma,q}$ of $F(\Sigma)$ associated with $q \in Q, \gamma \in \Gamma, q^+ = \delta(q,\gamma)$ is defined as
$$\hat{R}_{q^+,\gamma,q}(\hat{x}) = \hat{M}_{q^+,\gamma,q}\hat{x} + \hat{m}_{q^+,\gamma,q}$$
- The matrices $\hat{A}_q, \hat{B}_q, \hat{C}_q, \hat{M}_{q^+,\gamma,q}$, the polyhedron $\hat{\mathcal{P}}_q$, and vectors $\hat{a}_q, \hat{c}_q, \hat{m}_{q^+,\gamma,q}$ are defined as follows. Let $v_0, v_1, \dots, v_{d_q} \in \mathcal{P}_q$ be an affine basis of $\mathcal{P}_q$. Then there exists $v_{d_q+1}, \dots, v_{n_q} \notin \mathcal{P}_q$ such that $w_i = v_i - v_0$, $i = 1, \dots, n_q$, forms a basis of $\mathbb{R}^{n_q}$. Denote by W_q the linear span w_i, $i = 1, \dots, d_q$ and let S_q be the isomorphism $S_q : W_q \to \mathbb{R}^{d_q}$. Then $\mathbb{R}^{n_q}$ is the direct sum $W \oplus W_c$. Define the linear map $\Pi_q : \mathbb{R}^{n_q} \to \mathbb{R}^{d_q}$ by $\Pi_q(x) = S_q(x_1)$, where $x = x_1 + x_2$, $x_1 \in W_q$ and $x_2 \in W_c$. We identify Π_q with the corresponding matrix and denote by Π_q^{-1} the right inverse of Π_q. Then
$$\hat{A}_q = \Pi_q A_q \Pi_q^{-1}, \quad \hat{a}_q = \Pi_q a_q, \quad \hat{B}_q = \Pi_q B_q,$$
$$\hat{C}_q\Pi_q^{-1}, \; \hat{c}_q = c_q, \quad \hat{M}_{q^+,\gamma,q}v = \Pi_{q^+} M_{q^+,\gamma,q}\Pi_q^{-1},$$
$$\hat{m}_{q^+,\gamma,q} = \Pi_{q^+}(m_{q^+,\gamma,q}),$$
$$\hat{\mathcal{P}}_q = \bigcap_{\gamma\in\Gamma}\{\hat{x} \mid n_{q,\gamma}^T\Pi_q^{-1}\hat{x} \leq b_{q,\gamma}\}$$
- The initial state $\hat{h}_0$ of $F(\Sigma)$ is $\hat{h}_0 = (q_0, \Pi_q(x_0))$.

The above transformation preserves input-output behavior.

Theorem 5: With the notation of Definition 25, for any state $h = (q, x)$ of Σ, the state $\hat{h} = (q, \Pi_q(x))$ is a state of $F(\Sigma)$ and for any event generator G, $y_{\Sigma,G,h}$ exists if and only if $y_{F(\Sigma),G,\hat{h}}$ exists, and $y_{\Sigma,G,h} = y_{F(\Sigma),G,\hat{h}}$.

Next, we present the transformation of Σ to a linear **PAHS**.

*Definition 26 (**PAHS** to linear **PAHS**):* Define the *linear* **PAHS** $L(\Sigma)$ associated with Σ as follows.

- The set of discrete states and outputs, the discrete state-transition map and the discrete readout map $L(\Sigma)$ are the same as the corresponding items of Σ.
- For each $q \in Q$, the continuous state-space of $L(\Sigma)$ is $\bar{\mathcal{P}}_q \subseteq \mathbb{R}^{n_q+1}$ and the affine system is
$$\dot{\bar{x}}(t) = \bar{A}_q\bar{x}(t) + \bar{B}_q u(t)$$
$$y(t) = \bar{C}_q\bar{x}(t)$$
For each $\gamma \in \Gamma$, the reset map $\bar{R}_{q^+,\gamma,q}$, $q^+ = \delta(q,\gamma)$, of $L(\Sigma)$ is linear, i.e. $R_{q^+,\gamma,q}(x) = \bar{M}_{q^+,\gamma,q}x$. Here,
$$\bar{A}_q = \begin{bmatrix} A_q & a_q \\ 0 & 0 \end{bmatrix}, \; \bar{B}_q = \begin{bmatrix} B_q \\ 0 \end{bmatrix} \text{ and } \bar{C}_q = \begin{bmatrix} C_q & c_q \end{bmatrix}$$
$$\bar{M}_{q^+,\gamma,q} = \begin{bmatrix} M_{q^+,\gamma,q} & m_{q^+,\gamma,q} \\ 0 & 1 \end{bmatrix}$$
$$\bar{\mathcal{P}}_q = \bigcap_{\gamma\in\Gamma}\{\bar{x} \mid \bar{n}_{q,\gamma}^T\bar{x} \leq 0\}, \bar{n}_{q,\gamma} = \begin{bmatrix} n_{q,\gamma} \\ 1 \end{bmatrix}, \; \gamma \in \Gamma$$
- The initial state of $L(\Sigma)$ is $\bar{h}_0 = (q_0, (x_0, 1))$.

The intuition behind the construction of $L(\Sigma)$ is as follows. In order to encode the affine component of the continuous dynamics of Σ, for all discrete states an additional continuous state component is added. Hence, the state $(q, x, 1)$ of $L(\Sigma)$ corresponds to the state (q, x) of Σ.

Theorem 6: Using the notation of Definition 26, $L(\Sigma)$ is a linear **PAHS** . If Σ is full dimensional, then so is $L(\Sigma)$. For any state $h = (q, x)$ of Σ, $\bar{h} = (q, x, 1)$ is a state of $L(\Sigma)$, and for any event generator G, $y_{L(\Sigma),G,\bar{h}}$ exists if and only if $y_{\Sigma,G,h}$ exists, and $y_{L(\Sigma),G,\bar{h}} = y_{\Sigma,G,h}$.

D. Proof of Theorem 1 – 3

The proofs of Theorem 1 – 3 relies on the following result.

Theorem 7: A linear **PAHS** Σ is weakly observable if and only if the associated LHS H_Σ is observable.

The proof of Theorem 7 can directly be obtained by substituting the definition of $\mathbb{M}_q(w,v)$ and $O_{\Sigma,q}$ into Theorem 2 of [10]. *Proof:* [Proof of Theorem 2] Consider the LHS H_Σ associated with Σ. From Theorem 4 it follows that if Σ is observable, then H_Σ is observable, and hence by Theorem 7 Σ is weakly observable. ∎

Proof: [Proof of Theorem 1] Consider the linear **PAHS** $L(\Sigma)$ associated with Σ and let $H = H_{L(\Sigma)}$ be the LHS associated with $L(\Sigma)$. Notice that if Σ is full dimensional, then so is $L(\Sigma)$. Assume that for some $q \in Q$, there exists $O_{\Sigma,q} \neq 0$. Since $\mathcal{P}_q$ is a full dimensional, from Lemma 4 we get that there exists $x_1, x_2 \in \mathcal{P}_q$ such that $x_1 - x_2 \in O_{\Sigma,q}$.

Denote by W the subset of $\mathbb{R}^{n_q+1}$ formed by vectors of the form $(x^T, 0)^T$, $x \in \mathbb{R}^{n_q}$. Denote $O_{H,q} = O_{L(\Sigma),q}$. Note that $O_{H,q}$ is the observability subspace of H, as defined in [10], [9]. Note that $O_{H,q} \cap W = \{(z^T,0)^T \mid z \in O_{\Sigma,q}\}$. Hence, then $0 \neq ((x_1-x_2)^T, 0) \in O_{H,q} \cap W$. If we set $\hat{x}_i = (x_i^T, 1)^T$, $i = 1,2$, then $\hat{x}_1 - \hat{x}_2 = ((x_1-x_2)^T, 0) \in O_{H,q}$. Recall from [10] that $\hat{x}_1 - \hat{x}_2 \in O_{H,q}$ in fact implies that $y_{H,(q,\hat{x}_1-\hat{x}_2)} = 0$. By Lemma 3 the latter implies that $\upsilon_{H,\hat{h}_1} = \upsilon_{H,\hat{h}_2}$, $\hat{h}_i = (q, \hat{x}_i)$, $i = 1,2$. Then by Lemma 2, $y_{L(\Sigma),G,\hat{h}_1} = y_{L(\Sigma),G,\hat{h}_2}$ for any event generator G. Since by Theorem 6, $y_{\Sigma,G,h_i} = y_{L(\Sigma),G,\hat{h}_i}$, $i = 1,2$, where $h_i = (q, x_i)$, $i = 1,2$, we get that $y_{\Sigma,G,h_1} = y_{\Sigma,G,h_2}$ and $h_1 \neq h_2$. But this contradicts the observability of Σ. ∎

Proof: [Sketch of the proof of Theorem 3] Consider a **PAHS** Σ. Transform it to a full dimensional **PAHS** $\Sigma_1 = F(\Sigma)$. Transform Σ_1 to a linear (and full dimensional) **PAHS** $\Sigma_2 = L(\Sigma_1)$. By Theorem 6 and Theorem 5, if h_0 is the initial state of Σ and h_0^2 is the initial state of Σ_2, then for any event generator G, $y_{\Sigma,G,h_0} = y_{\Sigma_2,G,h_0^2}$.

Consider the LHS H associated with Σ_2. Assume that H is of the form (4) and Σ_2 is of the form (1). By [10], [9] we can transform H to an observable LHS H_o such that H and H_o realize the same input-output map, i.e. $\upsilon_{H,h_0^2} = \upsilon_{H_o,h_0^o}$, where h_0^2 is the initial state of H (which is the same as the initial state of Σ_2), and h_0^o is the initial state H_o.

We apply Definition 24 to H_o to obtain the **PAHS** Σ_o as $\Sigma_o = \Sigma_{H_o}$, i.e. Σ_o is the **PAHS** associated with H_o. Since $H_{\Sigma_o} = H_o$ and H_o is observable, we get that Σ_o is weakly observable. In addition, $\dim \Sigma_o = \dim H_o \leq \dim H = \dim \Sigma_2$ and $\dim \Sigma_2 \leq \dim(p, r+p)$ where $(p,r) = \dim \Sigma_1$, and $\dim \Sigma_1 \leq \dim \Sigma$. Moreover, if Σ is a linear **PAHS** , then so is Σ_1 and Σ_2, and then instead of Σ_2 we can take simply Σ_1, i.e. then $\dim \Sigma_2 = \dim \Sigma_1 \leq \Sigma$.

It is also easy to see that Σ_o is full dimensional, To this end, recall from [10], [9] the definition of an LHS morphism and recall that there exists a surjective LHS morphism $S = (S_D, S_C) : H \to H_o$. The existence and surjectivity of S implies that S_D is a map $S_D : Q \to Q_o$ and S_C is a linear map $S_C : \bigoplus_{q \in Q} \mathcal{X}_q \to \bigoplus_{q^o \in Q_o} \mathcal{X}^o_{q^o}$, where Q_o is the set of discrete states of H_o and $\mathcal{X}^o_{q^o}$ is the continuous state-spaces of H_o associated with discrete state $q_o \in Q_o$. In addition, it holds that $S_C(\mathcal{X}_q) \subseteq \mathcal{X}^o_{S_D(q)}$ and $\hat{C}_q S_C = \hat{C}^o_{S_D(q)}$, $q \in Q$, where $\hat{C}^o_{S_D(q)}$ is the output matrix of H_o associated with the discrete state $S_D(q)$. Hence, for all $q \in S_D^{-1}(q^o)$ and for any $e \in \Gamma$, if $\hat{n}^T_{q^o,\gamma}$ denotes the γth row of $\hat{C}^o_{q^o}$, then $\hat{n}^T_{q^o,\gamma} S_C = n^T_{q,\gamma}$, since $n^T_{q,\gamma}$ is the γth row of $\hat{C}_q$. This and the construction of the polyhedron $\mathcal{P}^o_{q^o}$ of Σ_o associated with the discrete state $q^o \in Q_o$, implies that $\mathcal{P}_{q^o}$ contains $S_C(\bigcup_{q \in S_D^{-1}(q^o)} \mathcal{P}_q)$. Since each $\mathcal{P}_q$ contains an affine basis of $\mathcal{X}_q$, $V = \bigcup_{q \in S_D^{-1}(q^o)} \mathcal{P}_q$ contains an affine basis of $W_{q^o} = \bigoplus_{q \in S_D^{-1}(q^o)} \mathcal{X}_q$. Due to subjectivity of S_C, the map $K : W_{q^o} \in x \mapsto S_C(x) \in \mathcal{X}^o_{q^o}$ is a surjective linear map. Hence, $S_C(V)$ contains an affine basis of $\mathcal{X}^o_{q^o}$, i.e. $\mathcal{P}^o_{q^o}$ is full-dimensional.

Notice that $\upsilon_{H,h_0} = \upsilon_{H_o,h_0^o}$. Moreover, H is the LHS associated with Σ_2 and hence by Lemma 2, $y_{\Sigma_2,G} = y_{\Sigma_2,G,h_0} = y_{\Sigma,G,h_0} = y_{\Sigma_o,G}$ for any event generator G. That is, Σ and Σ_o realizes the same input-output behavior. ∎

VI. Conclusions

We presented necessary conditions for observability of piecewise-affine hybrid systems, and observability reduction algorithm. The latter transforms a piecewise-affine system to a one which satisfies the necessary conditions. Future research is directed towards obtaining necessary and sufficient conditions for observability of piecewise-affine hybrid systems and an observability reduction algorithm.

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