# Chaos via Two-Valued Interval Maps in a Piecewise Affine Model Example for Hysteresis

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***Abstract*— A standard piecewise affine model of hysteresis in two dimensions is reconsidered. Periodic orbits without self-intersection are studied and, in terms of the two real parameters, their full bifurcation analysis is given. The main tool is a piecewise smooth, two-valued Poincaré mapping with only four points of discontinuities, the first return map with respect to the line connecting the two equilibria. Earlier, single-valued Poincaré mappings for the same model were associated with the switching lines and had an infinite number of discontinuities. The present paper ends with bifurcation curves responsible for larger/smaller supports of absolutely continuous invariant measures.**

KEYWORDS: **two-valued interval maps, hysteresis, chaos, switching dynamics**

## I. INTRODUCTION

The piecewise affine 2D pair of ordinary differential equations

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\sigma & 1\\ -1 & \sigma\end{pmatrix}\begin{pmatrix}x-1\\ y-\rho\end{pmatrix} \quad \text{if } x \geq -1 \text{ and } t \geq 0 \quad (1)$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\sigma & 1\\ -1 & \sigma\end{pmatrix}\begin{pmatrix}x+1\\ y+\rho\end{pmatrix} \quad \text{if } x \leq 1 \text{ and } t \geq 0 \quad (2)$$

is a well-known model example for hysteresis [4], [5], [6]. Here $\sigma$ and $\rho$ are real parameters. Note that system (1)–(2) is symmetric with respect to the origin and has two equilibrium points, $E^+ = (1, \rho)$ and $E^- = (-1, -\rho)$.

Throughout this paper, the Poincaré mapping for system (1)–(2) is defined as the first return map $\Pi$ associated with line $\mathcal{L} = \{\chi = (x, \rho x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$, the line connecting equilibria $E^+$ and $E^-$. By passing to coordinate representations,

$$\Pi : \mathcal{L} \to \mathcal{L} \quad , \quad \chi \mapsto \Pi(\chi) = (X(x), \rho X(x)) \, .$$

For $|x| \leq 1$, the local trajectory of system (1)–(2) through $(x, \rho x) \in \mathcal{L}$ is not uniquely defined. It can be governed first (i.e., on a time interval $[0, t^*(x))$ with some $t^*(x) > 0$) either by equation (1) or by equation (2). Consequently, mapping $x \mapsto X(x)$ has two branches defined on the intervals $[-1, \infty)$ and $(-\infty, 1]$, respectively. Thus

$$X(x) = \begin{cases} A(x) & \text{if } x \geq -1 \\ a(x) & \text{if } x \leq 1 \end{cases} \quad (3)$$

where functions $A : [-1, \infty) \to \mathbb{R}$ and $a : (-\infty, 1] \to \mathbb{R}$ are defined by the local solution dynamics. By symmetry, $A(x) = -a(-x)$ for each $x \geq 1$.

Since both equations (1) and (2) can be solved explicitly, it is elementary to check that function $A$ is defined for each $x \geq -1$. There is an exceptional point $q > -1$, however. The solution to equation (1) that starts at $Q = (q, \rho q) \in \mathcal{L}$ is tangent to the switching line $x = -1$ after time $\beta \in (0, \pi]$ at some point $G$ and then, excepting the special case $G = E^-$, it continues in two branches. Thus, function $A$ is two-valued at $q$, $A(q) = \{u_R, u_L\} \subset \mathbb{R}$ (and single-valued at each $x \in [-1, \infty) \setminus \{q\}$). The branches emerging from $G$ are governed by equation (1) resp. (2) and reach $\mathcal{L}$ in $U_R = (u_R, \rho u_R)$ resp. $U_L = (u_L, \rho u_L)$.

It is important to remark that solutions to system (1)–(2) can be started at points where the right-hand side is single-valued, i.e., at initial values $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2$ satisfying either $x_0 < -1$ or $x_0 > 1$. Any time a trajectory governed by equation (1) reaches the switching line $x = -1$ at a point different from $G$, it remains in the half-plane $\{(x, y) \in \mathbb{R}^2 \mid x \leq 1\}$ and it is governed by equation (2) for a while (until it hits the switching line $x = 1$ etc.).

Similarly, iterations of the Poincaré mapping $x \mapsto X(x)$ can be started at points where $X$ is single-valued. Clearly $(x_0, x_1) \in \text{Graph}(A)$ whenever $x_0 > 1$, and $(x_0, x_1) \in \text{Graph}(a)$ whenever $x_0 < -1$. Depending on $\sigma$ and $\rho$, the iteration may become multivalued if $x_k \in \{\pm 1, \pm q\}$. Any time $(x_{k-1}, x_k) \in \text{Graph}(A)$, then $x_{k+1} = A(x_k)$ if $x_k \geq -1$ and $a(x_k)$ if $x_k \leq -1$. Similarly, any time $(x_{k-1}, x_k) \in \text{Graph}(a)$, then $x_{k+1} = a(x_k)$ if $x_k \leq 1$ and $A(x_k)$ if $x_k \geq 1$. Thus, iterations of the Poincaré mapping $x \mapsto X(x)$ have to be understood as iterations on $\text{Graph}(a) \cup \text{Graph}(A)$. Actually, on their own right, iterations on $\text{Graph}(a) \cup \text{Graph}(A)$ can be started for $|x_0| \leq 1$ as well but they do not correspond to solution trajectories of system (1)–(2).

## II. DEFINING A TWO-VALUED POINCARÉ MAP

Now we return to the definition of the Poincaré map, more precisely, we pass to the underlying computations. (Of course everything depends on parameters $\sigma$ and $\rho$.) Without specifying the respective domains, solutions to equations (1)

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and (2) can be written as

$$\begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix} = e^{\sigma t} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 - 1 \\ y_0 - \rho \end{pmatrix} + \begin{pmatrix} 1 \\ \rho \end{pmatrix}$$

and

$$\begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix} = e^{\sigma t} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 + 1 \\ y_0 + \rho \end{pmatrix} - \begin{pmatrix} 1 \\ \rho \end{pmatrix},$$

respectively. Here, of course, $x(0,x_0,y_0) = x_0$ and $y(0,x_0,y_0) = y_0$. Existence and uniqueness of $G$ follows via considering the system of equations

$$x(\beta,q,\rho q) = -1 \quad , \quad \dot{x}(\beta,q,\rho q) = 0 \tag{4}$$

with unknowns $q > -1$ and $\beta \in (0,\pi]$. The solution is

$$\tan(\beta) = \frac{\sigma+\rho}{1-\sigma\rho} \quad \text{and} \quad q = 1 - \frac{2}{v(\beta)}$$

where $\beta = \frac{\pi}{2}$ if $\sigma\rho = 1$, $\beta = \pi$ if $\sigma + \rho = 0$ and function $v : [0,\pi] \to \mathbb{R}$ is defined by letting

$$v(\varphi) = e^{\sigma\varphi}(\cos(\varphi) + \rho\sin(\varphi)) \quad \text{for all} \quad \varphi \in [0,\pi]\,.$$

For brevity, we write $\alpha = \frac{\pi}{2} - \tan^{-1}(\rho)$. Note that

$$\beta = \begin{cases} \pi + \tan^{-1}(\rho) + \tan^{-1}(\sigma) & \text{if} \quad \sigma+\rho \le 0 \\ \tan^{-1}(\rho) + \tan^{-1}(\sigma) & \text{if} \quad \sigma+\rho > 0\,, \end{cases}$$

$$\sigma+\rho \le 0 \quad \Leftrightarrow \quad \pi - \alpha < \beta\,,$$
$$\sigma+\rho > 0 \quad \Leftrightarrow \quad \beta < \pi - \alpha\,.$$

Of course, as a reformulation of the requirement $x_0 = q > -1$, it remains to prove that $v(\beta) \ne 0$ and $\frac{1}{v(\beta)} < 1$. This is a by-product of the following result.

*Lemma 1:* The basic properties of function $v$ are as follows.
CASE $\sigma+\rho \le 0$ : Function $v$ is positive for $\varphi < \pi - \alpha$, negative for $\varphi > \pi - \alpha$, strictly decreasing for $\varphi \le \beta$, strictly increasing for $\varphi > \beta$.
CASE $\sigma+\rho > 0$ : Function $v$ is positive for $\varphi < \pi - \alpha$, negative for $\varphi > \pi - \alpha$, strictly increasing for $\varphi \le \beta$, strictly decreasing for $\varphi \ge \beta$.
In both cases, $v(0) = 1$, $v(\pi - \alpha) = 0$ and $\dot{v}(\varphi) \ne 0$ for $\varphi \ne \beta, \beta - \pi$.

*Proof:* The easy proof is omitted. (Observe that $\varphi \ne \beta - \pi$ makes sense only in the special case $\beta = \pi$.) □

A direct computation gives that

$$G = (-1, 2\sigma+\rho) \quad \text{and} \quad u_R = x(\pi,q,\rho q) = 1 + \frac{2e^{\sigma\pi}}{v(\beta)}\,.$$

For $\sigma + \rho \ge 0$, equation (2) has a trajectory segment in $(-\infty, 1]$ that starts at some $(-p, -p\rho) \in \mathcal{L}$ with $p > 1$ and ends in $E^+$ after time $\pi$. The intersection point between this trajectory and the switching line is denoted by $F$. It is easily shown that

$$p = 1 + \frac{2}{e^{\sigma\pi}} \quad \text{and} \quad F = \left(-1, -\rho + \frac{2}{e^{\sigma\alpha}\sin(\alpha)}\right)\,.$$

As for $u_L$, we distinguish three cases according to $u_L \le -1$, $-1 < u_L \le 1$, $1 < u_L$ or, equivalently, according to the relative position of the points $E^-$, $F$, $G$ on the swithing line $x = -1$:

CASE 1) $G \le E^- \;\Leftrightarrow\; \sigma+\rho \le 0$.
CASE 2) $E^- < G \le F \;\Leftrightarrow\; 0 < (\rho+\sigma)e^{\sigma\alpha}\sin(\alpha) \le 1$.
CASE 3) $E^- < F < G \;\Leftrightarrow\; (\rho+\sigma)e^{\sigma\alpha}\sin(\alpha) > 1$.

See Figures 1, 2 and 3, respectively. Recall that $Q = (q, \rho q)$ and define $P = (p, \rho p)$ [only in the last two cases], $R = (r, \rho r)$ [only in CASE 3] where $R \in \mathcal{L}$ is the starting point of the solution to equation (1) that, after time $\delta \in [0,\beta]$, arrives at $F$. Note that

$$\tan(\delta) = \frac{1}{(1+\rho^2)e^{\sigma\alpha}\sin(\alpha) - \rho} \quad \text{and} \quad r = 1 - \frac{2}{v(\delta)}\,.$$

Note also that we could not find any explicit formula for $u_L$ in CASE 3. In the first two cases, i.e.,

if $(\rho+\sigma)e^{\sigma\alpha}\sin(\alpha) \le 1$, then $u_L = 2(\rho+\sigma)e^{\sigma\alpha}\sin(\alpha) - 1$.

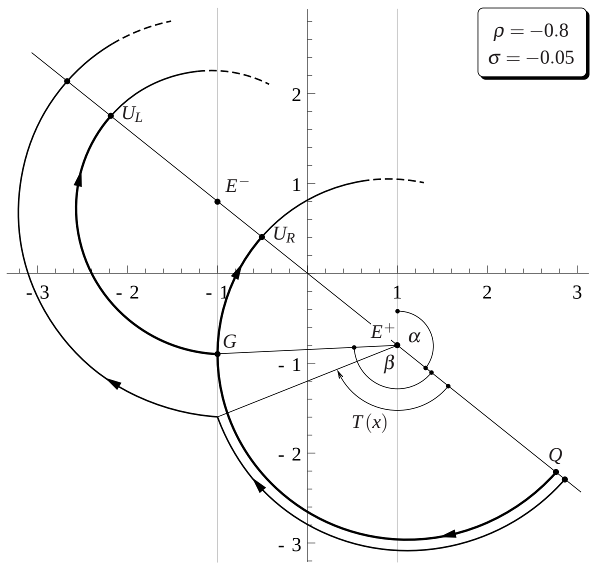


Fig. 1. CASE 1: $-1 < 1 < q$

For $x \in [-1,\infty)$ specified later (in a case-sensitive way), we introduce an affine function

$$L(x) = -e^{\sigma\pi}(x-1) + 1$$

and, with $T(x) \in [0,\pi]$ uniquely determined by equation

$$v(T(x)) = -\frac{2}{x-1} \tag{5}$$

(also in a case-sensitive way), a nonlinear function

$$N(x) = \frac{e^{\sigma\alpha}}{\sin(\alpha)} \cdot \frac{2\sin(T(x))}{\cos(T(x)) + \rho\sin(T(x))} - 1\,. \tag{6}$$

This makes a short description of function $A$ possible. As it is indicated in Figure 1, the description is based on the number of switches of the underlying trajectory segments of system (1)–(2) between $(x, \rho x) \in \mathcal{L}$ and $(A(x), \rho A(x)) \in \mathcal{L}$.

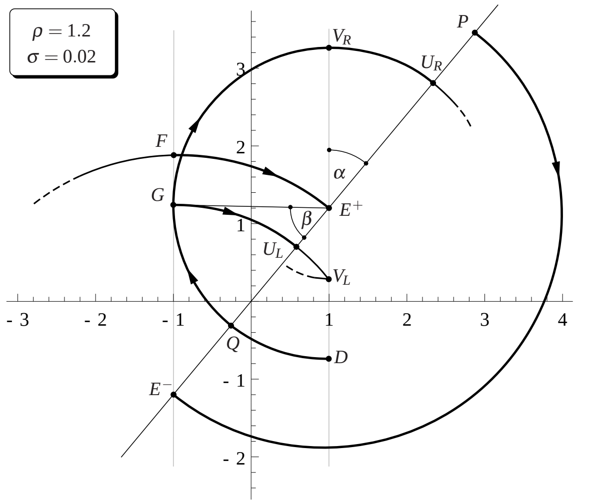


Fig. 2. CASE 2: $-1 < q < 1 < p$

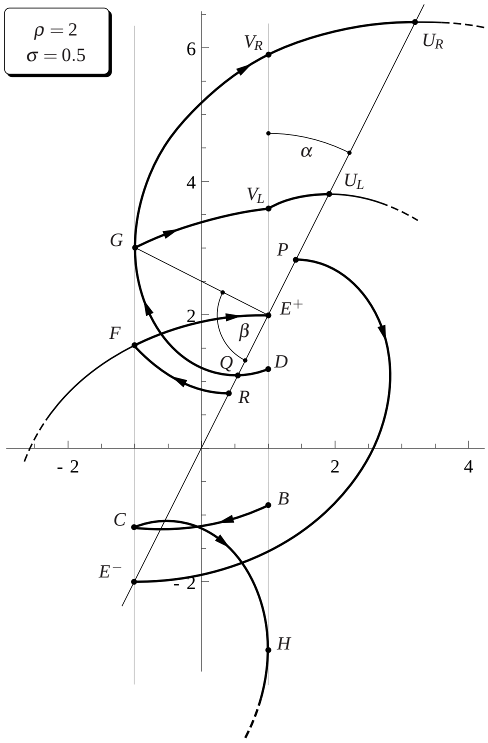


Fig. 3. CASE 3: $-1 < r < q < 1 < p$

*Number of switches: 0* $\quad \Leftrightarrow \quad A(x) = L(x)$

$$\left.\begin{array}{l}\text{CASE 1):}\\ \text{CASE 2):}\\ \text{CASE 3):}\end{array}\right\} \text{ if } \left\{\begin{array}{l} x \in [-1, q] \\ x \in [q, p] \\ x \in [q, p] \end{array}\right.$$

*Number of switches: 1* $\quad \Leftrightarrow \quad A(x) = N(x)$

$$\left.\begin{array}{l}\text{CASE 1):}\\ \text{CASE 2):}\\ \\ \text{CASE 3):}\\ \\ \end{array}\right\} \text{ if } \left\{\begin{array}{lll} T(x) \in (\pi-\alpha, \beta] & \text{and} & x \in [q, \infty) \\ T(x) \in [0, \beta] & \text{and} & x \in [-1, q] \\ T(x) \in (\pi-\alpha, \pi] & \text{and} & x \in [p, \infty) \\ T(x) \in [0, \delta] & \text{and} & x \in [-1, r] \\ T(x) \in (\pi-\alpha, \pi] & \text{and} & x \in [p, \infty) \end{array}\right.$$

*Number of switches: 2* $\quad \Leftrightarrow \quad A(x) = M(x)$

$$\text{CASE 3):} \quad \text{if} \quad x \in [r, q]$$

The details, especially those on the second nonlinear function $M$ are somewhat lengthy and, due to lack of space, are not presented here.

We need an auxiliary function $w : [0, \pi] \setminus \{\beta, \beta - \pi\} \to \mathbb{R}$ defined by letting

$$w(\varphi) = \frac{e^{\sigma\alpha}}{\sin(\alpha)} \cdot \frac{e^{\sigma\varphi}}{(\sigma+\rho)\cos(\varphi) - (1-\sigma\rho)\sin(\varphi)}$$

for all $\varphi \in [0, \pi] \setminus \{\beta, \beta-\pi\}$. Note that $\lim_{\varphi\to\beta} |w(\varphi)| = \infty$. On the other hand, $\beta - \pi \in [0, \pi]$ if and only if $\beta = \pi$ and then $\lim_{\varphi\to\beta-\pi} w(\varphi) = -\infty$.

*Lemma 2:* The basic properties of function $w$ are as follows.
CASE $\sigma + \rho \leq 0$ : Function $w$ is negative for $\varphi < \beta$, positive for $\varphi > \beta$, strictly increasing for $\varphi \leq \pi - \alpha$, strictly decreasing for $\pi - \alpha \leq \varphi < \beta$, and also for $\varphi > \beta$.
CASE $\sigma + \rho > 0$ : Function $w$ is positive for $\varphi < \beta$, negative for $\varphi > \beta$, strictly increasing for $\varphi < \beta$ and also for $\beta < \varphi \leq \pi - \alpha$, and strictly decreasing for $\varphi \geq \pi - \alpha$.
In both cases, $w(\pi - \alpha) = -e^{\sigma\pi}$ and $\dot{w}(\varphi) \neq 0$ for $\varphi \neq \pi - \alpha$.

*Proof:* The easy proof is omitted. □

*Theorem 1:* Function $A$ is piecewise smooth. In addition, whenever defined (including the endpoints of the respective intervals), functions $M$ and $N$ satisfy

$$\text{convexity: } M'', N'' > 0 \tag{7}$$

and, for $\sigma > 0$, they also satisfy

$$\text{expansivity: } |M'|, |N'| \geq \text{constant}(\sigma, \rho) > 1. \tag{8}$$

Further, functions $A|_{[-1,q]}$ and $A|_{[q,\infty)}$ are continuous. Differentiability may fail at the points $r \in (-1, q)$ and $p \in (q, \infty)$, respectively. Function $A|_{[q,\infty)}$ is strictly decreasing, $A|_{[-1,q]} = u_L$ in CASE 1, $A|_{[-1,q]} = u_R$ in CASES 2 and 3. On the other hand, function $A|_{[-1,q]}$ is strictly decreasing in CASE 1, and strictly increasing in CASES 2 and 3.

*Proof:* Nothing is hard although the inequalities require some care. The proof is given here only in the special cases which are actually needed in Theorem 2 below. Thus we restrict ourselves to function $N$ and to the intervals $[q, \infty)$ (in CASE 1) and $[p, \infty)$ (in CASES 2 and 3).

By using the derivative and then the square of equation (5), we obtain that

$$\dot{v}(T(x)) \cdot T'(x) = \frac{2}{(x-1)^2} = \frac{(v(T(x)))^2}{2}$$

It follows by direct computation that

$$N'(x) = \frac{e^{\sigma\alpha}}{\sin(\alpha)} \cdot \frac{2T'(x)}{(\cos(T(x)) + \rho\sin(T(x)))^2} = w(T(x)).$$

It is crucial that $T(x) \in (\pi - \alpha, \beta])$ in CASE 1 and $T(x) \in (\pi - \alpha, \pi]$ in CASES 2 and 3. Thus inequality (8) is a consequence of Lemma 2. Actually, by using a formulation valid in all the three CASES,

$$N'(x) < -e^{\sigma\pi} \qquad \text{whenever } x \geq \max\{q,p\}\,. \tag{9}$$

For later use, observe that $T(x) \to \pi - \alpha$ and thus

$$N'(x) \to w(\pi-\alpha) = -e^{\sigma\pi} \quad \text{ as } \; x \to \infty\,. \tag{10}$$

Since (10) is inconclusive for $\sigma = 0$, it is a must to note that

$$(N(x)+x) \to \frac{2\rho}{\sqrt{1+\rho^2}} \quad \text{ whenever } \; \sigma=0 \; \text{ and } \; x\to\infty\,. \tag{11}$$

In order to prove convexity, we start from

$$\dot{w}(\varphi) = \frac{e^{\sigma\alpha}}{\sin(\alpha)} \cdot \frac{(1+\sigma^2)v(\varphi)}{((\sigma+\rho)\cos(\varphi)-(1-\sigma\rho)\sin(\varphi))^2}\,,$$

and observe that the sign of

$$N''(x) = \dot{w}(T(x))\cdot T'(x) = \dot{w}(T(x))\cdot\frac{2}{(x-1)^2\cdot\dot{v}(T(x))}$$

is the same as the quotient of the signs of $v(T(x))$ and of $\dot{v}(T(x))$. Now Lemma 1 applies. □

## III. PERIODIC ORBITS WITHOUT SELF-INTERSECTION

*Definition 1:* For brevity, we say that a periodic orbit $\Gamma$ of system (1)–(2) is *large* if $\Gamma$ has no self-intersection points and it is not a subset of $\{(x,y)\in\mathbb{R}^2 \mid |x|\leq 1\}$. Similarly, we say that a periodic orbit $\gamma$ of system (1)–(2) is *small* if $\gamma$ has no self-intersection points and it is a subset of $\{(x,y)\in\mathbb{R}^2 \mid |x|\leq 1\}$.

### A. Large Periodic Orbits and their Bifurcations

*Theorem 2:* There exist continuous functions $\kappa_1:\mathbb{R}\to\mathbb{R}$ and $\kappa_2:(-\infty,0]\to\mathbb{R}$ with the properties that $\kappa_1(\sigma)<\kappa_2(\sigma)\leq 0$ for $\sigma\leq 0$, $\kappa_1(\sigma)<-\sigma$ for $\sigma>0$, and

$$\#\Gamma = 2 \;\Leftrightarrow\; \sigma<0\,,\ \kappa_1(\sigma)\leq\rho<\kappa_2(\sigma)$$

$$\#\Gamma = 1 \;\Leftrightarrow\; \begin{cases} \sigma<0\,,\ \rho=\kappa_2(\sigma) \;\text{ or }\; \rho<\kappa_1(\sigma) \\ \sigma=0\,,\ -1=\kappa_1(0)\leq\rho<\kappa_2(0)=0 \\ \sigma>0\,,\ \rho\geq\kappa_1(\sigma) \end{cases}$$

$$\#\Gamma = 0 \;\Leftrightarrow\; \begin{cases} \sigma<0\,,\ \rho>\kappa_2(\sigma) \\ \sigma=0\,,\ \rho<\kappa_1(0) \;\text{ or }\; \rho\geq\kappa_2(0) \\ \sigma>0\,,\ \rho\geq\kappa_1(\sigma) \end{cases}$$

Here $\#\Gamma$ stays for the number of large periodic orbits. See Figure 4.

*Proof:* Stability properties of large periodic orbits are discussed in Remark 1 below. Additional properties of functions $\kappa_1,\kappa_2$ are discussed in the proof itself. – The proof is divided into several steps.

*Step 1: A large periodic orbit is symmetric with respect to the origin.* It is enough to prove that, for $x>q$ (in CASE 1) and, for $x>p$ (in CASES 2 and 3), $x=a(A(x))$ implies that $A(x)=-x$. This will follow from the property that both $A|_{[q,\infty)}$ (in CASE 1) and $A|_{[p,\infty)}$ (in CASES 2 and 3) are strictly decreasing functions.

TABLE I

SEPARATION OF CASES IN THE SEARCH FOR LARGE PERIODIC ORBITS

| IF | AND | THEN | BY |
|---|---|---|---|
| $\sigma+\rho\leq 0\,,\sigma<0$ | $u_L<-q$ | $\#\Gamma=1$ | (7), (10) |
| | $u_L=-q$ | $\#\Gamma=2$ | (7), (10) |
| | $u_L>-q$ | $\#\Gamma=0,1,2$ | Step 4 |
| $\sigma+\rho<0\,,\sigma=0$ | $u_L<-q$ | $\#\Gamma=0$ | (7), (11) |
| | $u_L\geq-q$ | $\#\Gamma=1$ | (7), (11) |
| $\sigma+\rho\leq 0\,,\sigma>0$ | $u_L<-q$ | $\#\Gamma=0$ | (9) |
| | $u_L\geq-q$ | $\#\Gamma=1$ | (9) |
| $\sigma+\rho>0\,,\sigma\leq 0$ | | $\#\Gamma=0$ | Step 5 |
| $\sigma+\rho>0\,,\sigma>0$ | | $\#\Gamma=1$ | (9) |

If $1<q<x$ and $A(x)<-x<-1$, then $x=a(A(x))>a(-x)=-A(x)$, a contradiction. Similarly, if $1<q<x$ and $-x<A(x)<-1$, then $x=a(A(x))<a(-x)=-A(x)$, a contradiction.

Consider now an $x>p>1$ with $x=a(A(x))$. Since $A([1,p])\subset[-1,1]$, inequality $-p\leq A(x)<1$ leads to $x=a(A(x))\leq 1$, a contradiction. Thus $A(x)<-p$. As in the previous paragraph, we compose with $a$ from the left and see that both $A(x)<-x<-p$ and $-x<A(x)<-p$ are impossible. In fact, $A(x)<-x<-p$ implies that $x=a(A(x))>a(-x)=-A(x)$ and $-x<A(x)<-p$ implies that $-A(x)=a(-x)>a(A(x))=x$.

The final consequence is that large periodic orbits are characterized by the property $N(x)+x=0$ (with some $x>q$ (in CASE 1) and some $x>p$ (in CASES 2 and 3).

*Step 2: Direct implications of (7) and (9)–(11) on large periodic orbits.* We look for $\#\Gamma$, the number of large periodic orbits on a suitable decomposition of the parameter space $(\sigma,\rho)\in\mathbb{R}\times\mathbb{R}$. A quick analysis of the number of intersection points between the graphs of equations $y=N(x)$ and of $y=-x$ leads to Table 1 below.

*Step 3: Condition $\sigma+\rho\leq 0$, $u_L=-q$ and the resulting bifurcation curve $\mathcal{C}_1$.* We obtain by direct computation that

$$u_L>-q \quad\Leftrightarrow\quad W(\sigma,\rho)=e^{\sigma(\alpha+\beta)}\sin(\beta)-\sin(\alpha)<0\,,$$

$$W(0,\rho) = -\frac{\rho+1}{\sqrt{1+\rho^2}}<0 \quad\Leftrightarrow\quad -1<\rho\leq 0\,,$$

$$W(\sigma,0)=e^{\sigma(\frac{\pi}{2}+\beta)}\sin(\beta)-1<0 \quad \text{ whenever } \; \sigma<0\,,$$

$$W(\infty,\rho)=W(-\rho,\rho)=-\sin(\alpha)<0 \quad \text{ whenever } \; \rho\in\mathbb{R}$$

and, for $\sigma+\rho<0$,

$$\frac{\partial}{\partial\sigma}W(\sigma,\rho)=e^{\sigma(\alpha+\beta)}\sin(\beta)\left(\frac{3\pi}{2}+\tan^{-1}(\sigma)+\frac{1}{\sigma+\rho}\right).$$

Clearly $\frac{\partial}{\partial\sigma}W(\sigma,\rho)>(=,<)0$ if and only if $\rho<(=,>)\rho(\sigma)$ where function $\rho:\mathbb{R}\to\mathbb{R}$ is defined by letting

$$\rho(\sigma)=-\sigma-\frac{1}{\frac{3\pi}{2}+\tan^{-1}(\sigma)} \quad \text{ for each } \; \sigma\in\mathbb{R}\,.$$

Since $\rho'(\sigma) < -1$ and $\rho(\sigma) < -\sigma$, it follows that mapping $\sigma \mapsto \rho(\sigma)$ defines a self-diffeomorphism of $\mathbb{R}$ with inverse $\rho \mapsto \sigma(\rho)$ and $\text{argmax}_{\sigma \leq -\rho} W(\cdot, \rho) = \sigma(\rho)$ for each $\rho \in \mathbb{R}$. On the other hand, substituting $W(\sigma, \rho) = 0$ in

$$\frac{\partial}{\partial \rho} W(\sigma, \rho) = (e^{\sigma(\alpha+\beta)} \cos(\beta) + \cos(\alpha)) \cdot \frac{1}{1+\rho^2} ,$$

we obtain that

$$\frac{\partial}{\partial \rho} W(\sigma, \rho) \Big|_{W(\sigma,\rho)=0} = \frac{\sin(\alpha+\beta)}{\sin(\beta)} \cdot \frac{1}{1+\rho^2} < 0 .$$

By elementary implicit function arguments, we conclude there exists a smooth function $\kappa_1 : \mathbb{R} \to \mathbb{R}$ such that $W(\sigma, \rho) = 0$ if and only if $\rho = \kappa_1(\sigma)$ and, with some $\sigma^* > 0$, $(\kappa_1)'(\sigma) < (=, >)\ 0$ if and only if $\sigma > (=, <)\ \sigma^*$. In particular, the maximum of $\kappa_1$ is attained at $\sigma = \sigma^*$ and $\rho^* = \kappa_1(\sigma^*) = \rho(\sigma^*)$ satisfies $-1 < \rho^* < 0$. Computer simulation gives $(\sigma^*, \rho^*) = (0.3315..., -0.5301...)$. It is readily seen that $\kappa_1(\sigma) + \sigma \to 0$ as $\sigma \to \infty$.

Finally, we take $\mathcal{C}_1 = \text{Graph}(\kappa_1)$.

*Step 4: The hard parameter set $\mathcal{H} = \{(\sigma, \rho) \in \mathbb{R}^2 \mid \sigma + \rho \leq 0,\ \sigma < 0,\ u_L > -q\}$ and the resulting bifurcation curve* $\mathcal{C}_2$. We focus our attention to the case $\#\Gamma = 1$. By using (5) and (7), we see that the defining system of equations for $\Gamma$ is

$$N(x) + x = 0 \ , \quad v(T(x)) = -\frac{2}{x-1} \ , \quad N'(x) = -1. \tag{12}$$

Here $\Gamma \cap \mathcal{L} = \{(x, \rho x), (N(x), \rho N(x)\}$ and $T(x) \in (\pi - \alpha, \beta]$. Recall that $N'(x) = w(T(x))$.

A miracle happens. System (12) can be solved explicitly. In fact, the first two equations result in

$$\frac{e^{\sigma\alpha}}{\sin(\alpha)}\, e^{\sigma T} \sin(T) = 1 \tag{13}$$

and the last equation can be rewritten as

$$\frac{e^{\sigma\alpha}}{\sin(\alpha)} \cdot \frac{e^{\sigma T}}{(\sigma+\rho)\cos(T) - (1-\sigma\rho)\sin(T)} = -1 .$$

Hence

$$(\sigma+\rho)\sin(T)\cos(T) - (1-\sigma\rho)\sin^2(T) = -1 .$$

But this is a second order algebraic equation for $\tan(T)$. The solutions are $\tan(T)_1 = -\frac{1}{\sigma}$ with

$$T = \alpha + \beta - \pi \quad \text{and} \quad x = 1 + \frac{e^{\sigma\alpha}}{\sin(\alpha)} \cdot \frac{2}{\sigma - \rho}$$

(implying also $\sigma > \rho$ via $1 < q \leq x$) and $\tan(T)_2 = -\frac{1}{\rho}$ with $T = \pi - \alpha$, a contradiction.

Rearranging equation (13), we define function $Z$ with domain of definition $\{(\sigma, \rho) \in \mathbb{R}^2 \mid \sigma < 0 \text{ and } \rho \in (\sigma, 0)\}$ by letting

$$Z(\sigma, \rho) = e^{\sigma(T+\alpha)} \sin(T) - \sin(\alpha) .$$

Direct computation gives that

$$Z(\infty, \rho) = Z(0^-, \rho) = -\sin(\alpha) < 0 \quad \text{whenever} \quad \rho < 0$$

and, on the entire domain of function $Z$,

$$\frac{\partial}{\partial \sigma} Z(\sigma, \rho) = (T+\alpha) e^{\sigma(T+\alpha)} \sin(T) > 0 \ , \quad \frac{\partial}{\partial \rho} Z(\sigma, \rho)$$

$$= \frac{1}{1+\rho^2} \left\{ e^{\sigma(T+\alpha)} \left( -\frac{\sigma}{\sqrt{1+\sigma^2}} \right) + \frac{\rho}{\sqrt{1+\rho^2}} \right\} < 0 .$$

By elementary implicit function arguments, we conclude there exists a smooth function $\kappa_2 : (-\infty, 0) \to \mathbb{R}$ such that $Z(\sigma, \rho) = 0$ if and only if $\rho = \kappa_2(\sigma)$. By letting $\kappa_2(0) = 0$, function $\kappa_2$ is continuously extended to the closed half-line $(-\infty, 0]$. This is possible because $Z(0^-, 0^-) = 0$. Since $\frac{\partial}{\partial \sigma} Z(0^-, 0^-) = \pi > 0$ and $\frac{\partial}{\partial \rho} Z(0^-, 0^-) = 0$, we have also that $(\kappa_2)'(\sigma) \to \infty$ as $\sigma \to 0^-$.

Finally, we take $\mathcal{C}_2 = \text{Graph}(\kappa_2)$.

By the construction, $\#\Gamma = 1$ in the hard parameter set $\mathcal{H}$ if and only if $(\sigma, \rho) \in \mathcal{C}_2$. Now we make use of the results on $\mathcal{H}$ in Step 2 (actually, in the first row of Table 1) and on $\mathcal{C}_1$ in Step 3. They imply that $\mathcal{C}_2$ separates $\mathcal{H}$ in two arcwise connected components, one on the 'left-hand', one on the 'right-hand' side of $\mathcal{C}_2$. On each of these components, $\#\Gamma$ has to be a constant, either 0 or 2. It is readily checked that $\#\Gamma = 0$ on the 'left-hand', and $\#\Gamma = 2$ on the 'right-hand' side of $\mathcal{C}_2$ in $\mathcal{H}$.

*Step 5: The remaining $\sigma + \rho > 0$, $\sigma \leq 0$ case is easy.* We have to prove only the geometrically trivial chain of inequalities

$$x - 1 \geq \frac{\rho - Y(1)}{\sqrt{1+\rho^2}} > \frac{-\rho - Y(-1)}{\sqrt{1+\rho^2}} \geq -(N(x)+1)$$

where $\{(x, Y(x)) \in \mathbb{R}\times\mathbb{R} | -1 \leq x \leq 1\}$ is a parametrization of the crucial solution segment to equation (1) through $(x, \rho x) \in \mathcal{L}$ with $x > p$. The side inequalities are implied by $\sigma \leq 0$. The middle inequality combines $\rho \geq 0$ and $Y(-1) > Y(1)$ . The $(x, Y(x))$ parametrization is possible because $\dot{x} = \sigma(x-1) + (y-\rho) < 0$ whenever $-1 \leq x \leq 1$. Similarly, $\dot{y} = -(x-1) + \sigma(y-\rho) \geq (-1+\sigma\rho)(x-1) > 0$ and thus $Y'(x) = \frac{\dot{y}}{\dot{x}} < 0$ whenever $-1 \leq x < 1$. Integrating from $x = -1$ to $x = 1$, we arrive at the remaining inequality $Y(-1) > Y(1)$. Hence $N(x) + x > 0$ for each $x > p$ and therefore, $\#\Gamma = 0$.

*Step 6: The end of the proof.* Now all the desired results follow immediately by Table 1 when interpreted with respect to the curves $\mathcal{C}_1$ and $\mathcal{C}_2$. □

*Remark 1:* Fix $\rho = \rho_0 < -1$. At $(\sigma_0, \rho_0) \in \mathcal{C}_2$, system (1)–(2) undergoes a saddle-node bifurcation for periodic orbits. The emerging unstable and stable periodic orbits disappear at $(\sigma_-, \rho_0) \in \mathcal{C}_1$, $\sigma_- < 0$ [discontinuity-based grazing bifurcation at the switching lines with symmetry] and $(0, \rho_0)$ [coalescence with the point at infinity], respectively. Increasing parameter $\sigma$ a bit further, the unstable periodic orbit is reborn at $(\sigma_+, \rho_0) \in \mathcal{C}_1$, $\sigma_+ > 0$. Any time $(\sigma, \rho_0) \in \mathcal{C}_1$, point $G$ is situated on the corresponding large periodic orbit. Though $\beta$ is discontinuous at the line $\sigma + \rho = 0$, the dependence of $\Gamma$ on the parameter pair $(\sigma, \rho)$ is continuous.

For $(\sigma, \rho_0) \in \mathbb{R}^2$, $\sigma_- < \sigma < 0$, the (uniquely defined) large periodic orbit is stable and equilibria $E^+$ and $E^-$ are also stable. This is not a contradiction to the Poincaré index theorem, because the unreachable set is nonempty.

*B. Small Periodic Orbits and their Bifurcations*

For $\sigma + \rho > 0$, set

$$D = \left(1, \rho - \frac{2e^{-\sigma(\alpha+\beta)}}{\sin(\alpha+\beta)}\right) \quad , \quad C = -D \ , \ H = -G \, .$$

Point $D < E^+$ on the switching line $x = 1$ is chosen in such a way that the solution to equation (1) starting at $D$ passes through $Q$ and, after the total time $\alpha + \beta$, arrives at $G$.

Let $Y(1) \leq D$. The notation already anticipates that the solution to equation (1) through the point $(1, Y(1))$ can be represented as $(x, Y(x))$ near $x = 1$ and actually, on the interval $[-1, 1]$.

*Lemma 3:* Assume that $Y(1) < \tilde{Y}(1) \leq D$. Then $Y(-1) < \tilde{Y}(-1) \leq G$ and

$$\tilde{Y}(1) - Y(1) < \tilde{Y}(-1) - Y(-1) \, . \qquad (14)$$

*Proof:* The first assertion is obvious. Similarly, we have $Y(x) < \tilde{Y}(x)$ for all $x \in [-1, 1]$.

The second assertion is obtained now by integrating

$$\begin{aligned} \tilde{Y}'(x) &= \frac{-(x-1) + \sigma(\tilde{Y}(x) - \rho)}{\sigma(x-1) + (\tilde{Y}(x) - \rho)} \\ &< \frac{-(x-1) + \sigma(Y(x) - \rho)}{\sigma(x-1) + (Y(x) - \rho)} = Y'(x) \, , \end{aligned}$$

from $-1$ to 1, itself a consequence of inequality

$$\frac{\partial}{\partial y}\left(\frac{-(x-1) + \sigma(y-\rho)}{\sigma(x-1) + (y-\rho)}\right) = \frac{(x-1)(1+\sigma^2)}{(\sigma(x-1) + (y-\rho))^2} < 0$$

(valid for and the pair $(x, y) \in \mathbb{R}^2$ laying under the solution segment of equation (1) with endpoints $D$ and $G$). □

*Theorem 3:* There exist a continuous function $\kappa_3 : \mathbb{R} \to \mathbb{R}$ with the properties that $\kappa_3(\sigma) > -\sigma$ for $\sigma \in \mathbb{R}$, and

$$\begin{aligned} \#\gamma = 1 \quad &\Leftrightarrow \quad \rho \geq \kappa_3(\sigma) \\ \#\gamma = 0 \quad &\Leftrightarrow \quad \rho < \kappa_3(\sigma) \, . \end{aligned}$$

Here $\#\gamma$ stays for the number of small periodic orbits. See Figure 4. A small periodic orbit is always unstable.

*Proof:* Additional properties of function $\kappa_3$ are discussed in the proof itself. – The proof is divided into several steps.

*Step 1: A small periodic orbit is symmetric with respect to the origin.* Let $(1, \tilde{Y}(1))$ and $(1, Y(1))$ with $Y(1) < \tilde{Y}(1) \leq D$ belong to small periodic orbits. Recall that system (1)–(2) is symmetric with respect to the origin. Thus $C \leq Y(-1) < \tilde{Y}(-1)$ and a twofold application of Lemma 3 results in

$$\tilde{Y}(1) - Y(1) < \tilde{Y}(-1) - Y(-1) < \tilde{Y}(1) - Y(1)$$

a contradiction. This implies instability and uniqueness for small periodic orbits if any. Together with $(1, Y(1))$, also $(1, \tilde{Y}(1)) = (1, -Y(-1))$ belongs to a small periodic orbit. The desired symmetry property $-Y(-1) = Y(1)$ follows from uniqueness already established.

*Step 2: Conditions for existence and nonexistence of small periodic orbits.* By an immediate phase portrait analysis, there are no small periodic orbits if $G \leq E^-$. The same argument extends to the case $C > G > E^-$. Recall that $G > E^-$ is equivalent to $\sigma + \rho > 0$.

From now on, assume that $G > E^-$ and $C \leq G$. See Figures 2 and 3. Point $B < D$ on the switching line $x = 1$ is chosen in such a way that the solution to equation (1) starting at $B$ arrives at $C$. In view of Lemma 3, $H < B \leq D$. On the other hand, let $V_L$ denote the (first) intersection point of the switching line $x = 1$ and the solution to equation (2) starting at $G$. Applying Lemma 3 again, $V_L - H > G - C = D - H$ which simplifies to $V_L > D$. Since $H < B$ and $V_L > D$, a standard Bolzano–Darboux argument implies (regardless of $V_L < E^+$ or $V_L \geq E^+$) that, for some $B < Y^* < D$, $Y^*$ belongs to a small periodic orbit.

*Step 3: Inequalities $G > E^-$, $C \leq G$ and the resulting bifurcation curve $\mathcal{C}_3$.* Recall that $G > E^-$ is equivalent to $\sigma + \rho > 0$. We obtain by direct computation that with the same function—but on a different subset of the parameter space: a miracle again—introduced in Step 3 of proving the previous theorem,

$$C \leq G \quad \Leftrightarrow \quad W(\sigma, \rho) = e^{\sigma(\alpha+\beta)} \sin(\beta) - \sin(\alpha) > 0 \, .$$

Now we have for $\rho \in \mathbb{R}$ that $W(-\rho^+, \rho) = -\sin(\alpha) < 0$ and $W(\infty, \rho) = \infty$. On the other hand, for $\sigma + \rho > 0$, both

$$\frac{\partial}{\partial \sigma} W(\sigma, \rho) = e^{\sigma(\alpha+\beta)} \sin(\beta) \left(\frac{\pi}{2} + \tan^{-1}(\sigma) + \frac{1}{\sigma + \rho}\right)$$

$$\text{and} \qquad \left.\frac{\partial}{\partial \rho} W(\sigma, \rho)\right|_{W(\sigma,\rho)=0} = \frac{\sin(\alpha+\beta)}{\sin(\beta)} \cdot \frac{1}{1+\rho^2}$$

are positive. By elementary implicit function arguments, we conclude there exists a smooth and strictly decreasing function $\kappa_3 : \mathbb{R} \to \mathbb{R}$ such that $W(\sigma, \rho) < (=, >) \ 0$ if and only if $\rho < (=, >) \ \kappa_3(\sigma)$. It is readily seen that $\kappa_3(\sigma) > -\sigma$ for each $\sigma \in \mathbb{R}$ and $\kappa_3(\sigma) + \sigma \to 0$ as $\sigma \to \infty$. Asymptotic expansion gives that $\kappa_3(\sigma) > -(e-1)\sigma$ for each $\sigma \in \mathbb{R}$ and $\kappa_3(\sigma) + (e-1)\sigma \to 0$ as $\sigma \to -\infty$.

Finally, we take $\mathcal{C}_3 = \text{Graph}(\kappa_3)$. □

*Remark 2:* It is easy to point out that $C \leq G$ is a consequence of $G > F$. Suppose not. Then $C > G > F > E^-$ and $H > D$. By an elementary Bolzano–Darboux argument, it follows that the solution segment of equation (2) connecting $C$ and $H$ cuts the solution segment of equation (2) connecting $F$ and $E^+$, a contradiction. Thus $\#\gamma = 1$ whenever $(\rho + \sigma)e^{\sigma\alpha} \sin(\alpha) > 1$, the defining inequality for CASE 3. It is not hard but rather lengthy to reformulate inequality $(\rho + \sigma)e^{\sigma\alpha} \sin(\alpha) > 1$ as the pair of inequalities $0 < \sigma$ and $\rho > \kappa_4(\sigma)$ where $\kappa_4 : (0, \infty) \to \mathbb{R}$ is a smooth and strictly decreasing function with the properties that $\kappa_4(\sigma) > \kappa_3(\sigma)$ for each $\sigma > 0$, $\kappa_4(\sigma) \to \infty$ as $\sigma \to 0^+$ and $\kappa_4(\sigma) + \sigma \to 0$ as $\sigma \to \infty$.

## IV. BIFURCATIONS BETWEEN DIFFERENT TYPES OF CHAOS

From now on, assume that $\sigma > 0$ and recall the expansivity property established in (8). Expansive, piecewise smooth mappings in one dimension are known to have a great number of chaotic properties on positively invariant compact intervals [1]. From the single-valued to the two-valued case, results of this type are extended in [2], [3]. Here we restrict ourselves to investigate, in terms of parameters $\sigma$ and $\rho$, the existence of invariant intervals for system (1)–(2).

*Definition 2:* For brevity, we say that a nontrivial and compact interval $\mathbb{I} \subset \mathbb{R}$ is *small invariant* if it is positively invariant with respect to exactly one of the defining equations of system (1)–(2) and no other interval containing $\mathbb{I}$ has this property. Similarly, $\mathbb{J}$ is *large invariant* if it is positively invariant with respect to both individual equations of system (1)–(2) and no other interval containing $\mathbb{J}$ has this property.

Recall that $\sigma > 0$. We now already from Theorem 2 that, excepting the case $\sigma + \rho \leq 0$, $u_L < -q$, system (1)–(2) has a uniquely defined large periodic orbit $\Gamma$, $\Gamma \cap \mathcal{L} = \{\pm S\}$ with some $S = (s, \rho s)$, $s > 1$, and $N(s) + s = 0$. In addition, $s > q$ in the special case $\sigma + \rho \leq 0$, $u_L \geq -q$. On the other hand, $s > p$ whenever $\sigma + \rho > 0$.

*Lemma 4:* Recall that $\sigma > 0$. Then $u_R(\sigma, \rho) \geq u_L(\sigma, \rho)$.

*Proof:* Clearly $u_R \geq -1 \geq u_L$ in CASE 1 and $u_R > 1 \geq u_L$ in CASE 2. CASE 3 is highly nontrivial. What we prove is $v_R > v_L$ where $V_R = (1, v_R)$ and $V_L = (1, v_L)$ are the intersection points between the switching line $x = 1$ and the trajectories of equation (1) and equation (2) starting at the point $G > F$, respectively. (This is equivalent to $u_R > u_L$ whenever $G > F$.) By direct computation,

$$v_R + \rho = 2\rho + 2\mu(\sigma) = 2\rho + 2e^{\sigma \tan^{-1}(\frac{1}{\sigma})}\sqrt{1 + \sigma^2}\,,$$

$$v_L + \rho = \frac{2}{\tan(\tau(\rho, \sigma))}$$

where $\tau = \tau(\sigma, \rho)$ is the unique solution to equation

$$(\sigma + \rho)e^{\sigma\tau}\sin(\tau) = 1 \quad , \qquad 0 < \tau < \alpha\,. \qquad (15)$$

Since functions $\varphi \to e^{\sigma\varphi}\sin(\varphi)$ and $\varphi \to \tan(\varphi)$ are increasing on the interval $(0, \frac{\pi}{2})$, inequality $v_R > v_L$ is equivalent to $\tau(\sigma, \rho) > \tan^{-1}(\frac{1}{\rho + \mu(\sigma)})$ which, in turn, is equivalent to

$$e^{\sigma\tau}\sin(\tau) > e^{\sigma \tan^{-1}(\frac{1}{\rho+\mu(\sigma)})} \cdot \sin(\tan^{-1}(\frac{1}{\rho + \mu(\sigma)}))\,.$$

Multiplying both sides by $(\sigma + \rho)$, we conclude by using (15) that $v_R > u_L$ is equivalent to

$$1 > e^{\sigma \tan^{-1}(\frac{1}{\rho+\mu(\sigma)})} \cdot \frac{\sigma + \rho}{\sqrt{1 + (\rho + \mu(\sigma))^2}}\,.$$

Since $\mu(\sigma) > c\sigma$ for some fixed $c \geq 2$ (computer simulation suggests that $c$ can be chosen for $e = 2.71\ldots$.) and thereby

$$e^{\sigma \tan^{-1}(\frac{1}{\rho+\mu(\sigma)})} \cdot \frac{\sigma + \rho}{\sqrt{1 + (\rho + \mu(\sigma))^2}} < e^{\frac{\sigma}{\rho + c\sigma}} \cdot \frac{\sigma + \rho}{\rho + c\sigma}\,,$$

it is enough to show that

$$1 > e^{\frac{1}{\kappa + c}} \cdot \frac{1 + \kappa}{\kappa + c} \qquad \text{whenever} \quad \kappa = \frac{\rho}{\sigma} > -1\,. \qquad (16)$$

But the function on the right-hand side of inequality (16) is increasing for $\kappa > -1$ and tends to 1 as $\kappa \to \infty$. $\square$

Our final result improves those in [4] on different types of chaos. The bifurcation curves $\mathcal{C}_4$ and $\mathcal{C}_5$ are responsible for what is usually called the transition between single-spiral and double-spiral chaotic attractors as well as the transition between double-spiral chaotic attractors and chaotic repellers in system (1)–(2), respectively. See [3] for more details.

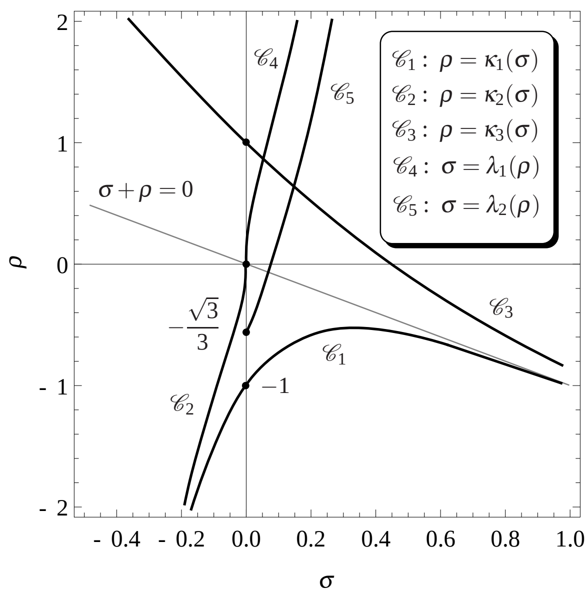


Fig. 4. The nontrivial bifurcation curves. (There is also a sixth, a trivial one: $\sigma = 0$.)

*Theorem 4:* Recall that $\sigma > 0$. Then there exist continuous mappings $\lambda_1 : [0, \infty) \to \mathbb{R}$ and $\lambda_2 : [-\frac{\sqrt{3}}{3}, \infty) \to \mathbb{R}$ with the properties that $\lambda_2(\rho) > \lambda_1(\rho) > \lambda_1(0) = 0$ for $\rho > 0$, $\lambda_2(\rho) > \lambda_2(-\frac{\sqrt{3}}{3}) = 0$ for $-\frac{\sqrt{3}}{3} < \rho < 0$, and,

$$\#\mathbb{I} > 0 \quad \Leftrightarrow \quad \begin{cases} 0 < \rho\,,\ 0 < \sigma \leq \lambda_1(\rho) \\ \text{and then} \ \ \mathbb{I} = [-1, p], \mathbb{I} = [-p, 1] \end{cases}$$

$$\#\mathbb{J} > 0 \quad \Leftrightarrow \quad \begin{cases} -\frac{\sqrt{3}}{3} < \rho\,,\ 0 < \sigma \leq \lambda_2(\rho) \\ \text{and then} \ \ \mathbb{J} = [-s, s] \end{cases}$$

Here $\#\mathbb{I}$ and $\#\mathbb{J}$ stay for the number of small and large invariant intervals, respectively. See Figure 4.

*Proof:* Additional properties of functions $\lambda_1$, $\lambda_2$ are discussed in the proof itself. – The proof consists of three steps.

*Step 1: Small invariant intervals and the resulting bifurcation curve $\mathcal{C}_4$.* If $\sigma + \rho \leq 0$, then the existence of small invariant intervals is excluded by property

$$A(-1) > q \quad \Leftrightarrow \quad e^{\sigma(\pi + \beta)} > \sin(\alpha) \cdot \sqrt{1 + \sigma^2}\,,$$

a direct consequence of inequality $e^{\sigma\pi} > \sqrt{1+\sigma^2}$ valid for each $\sigma > 0$. Consider now the $\sigma + \rho > 0$, $\sigma > 0$ case, and recall the monotonicity properties of function $A$ established in Theorem 2. In view of Lemma 4, $\mathbb{I} \subset [-1, \infty)$ is a small invariant interval with respect to equation (1) $\Leftrightarrow \mathbb{I} = [-1, p]$ and $u_R \leq p$. When applied to equation

$$\chi(\sigma, \rho) = u_R(\sigma, \rho) - p(\sigma) = 0 \, ,$$

existence and properties of the desired function $\lambda_1$ follow from the implicit function theorem. In fact, $\chi(0, \rho) = 2\cos(\beta) - 2 < 0$ for $\rho > 0$, $\chi(0^+, \rho) = 0$ for $\rho = 0$, and $\chi((-\rho)^+, \rho) = 2(e^{-\rho\pi} - e^{\rho\pi}) > 0$ for $\rho < 0$. On the other hand, $\chi(\infty, \rho) = \infty$ for each $\rho \in \mathbb{R}$, and $\frac{\partial}{\partial\sigma}\chi(\sigma, \rho) > 0$ whenever $\sigma + \rho > 0$. This last inequality is a combination of

$$\frac{\partial}{\partial\sigma} u_R(\sigma, \rho) = 2(\pi - \beta) e^{\sigma(\pi - \beta)} \frac{\sqrt{1+\sigma^2}}{\sqrt{1+\rho^2}} > 0 \qquad (17)$$

and $\frac{\mathrm{d}}{\mathrm{d}\sigma} p(\sigma) = 2\pi e^{-\sigma\pi} > 0$. Implicit differentiation leads to

$$\frac{\partial}{\partial\rho}\lambda_1(\rho) = \left. \frac{\sigma + \rho}{(2\pi - \beta)(1+\rho^2)} \right|_{\sigma = \lambda_1(\rho)} > 0 \quad \text{for} \quad \rho > 0$$

and $\frac{\partial}{\partial\rho}\lambda_1(0) = 0$. We take $\mathcal{C}_4 = \text{Graph}(\lambda_1)$.

*Step 2: The first partial derivatives of function $s$.* Differentiating formulas (5) and (6) with respect to $\sigma$, we obtain by using Lemmas 1 and 2 (in the $\sigma + \rho \leq 0$, $u_L \geq -q$, $T(x) \in (\pi - \alpha, \beta]$ and the $\sigma + \rho > 0$, $T(x) \in (\pi - \alpha, \pi]$ special cases) that both

$$\frac{\partial}{\partial\sigma} T = -\frac{(\cos(T) + \rho \sin(T)) \cdot T}{(\sigma + \rho)\cos(T) - (1 - \sigma\rho)\sin(T)}$$

and

$$\frac{\partial}{\partial\sigma} N = \frac{2e^{\sigma\alpha}}{\sin(\alpha)} \cdot \frac{\alpha(\cos(T) + \rho\sin(T))\sin(T) + \frac{\partial}{\partial\sigma}T}{(\cos(T) + \rho\sin(T))^2}$$

are negative. Thus $N(\sigma, \rho, s(\sigma, \rho)) + s(\sigma, \rho) = 0$ implies via inequality (9) that

$$\frac{\partial}{\partial\sigma} s = -\frac{\partial}{\partial\sigma} N(s)/(1 + N'(s)) < 0 \, . \qquad (18)$$

A similar computation yields by using (13) that

$$\frac{\partial}{\partial\rho} T = -w(T) \cdot \sin^2(T) > 0$$

and, with $m(T) = \frac{\sin(T)}{\cos(T) + \rho\sin(T)}$,

$$\frac{\partial}{\partial\rho} N = \frac{2e^{\sigma\alpha}}{\sin(\alpha)} \left( \frac{\rho - \sigma}{1+\rho^2} \, m(T) - (w(T) + 1) m^2(T) \right)$$

which is positive if $\rho \leq \sigma$. Here again, the conclusion is that

$$\frac{\partial}{\partial\rho} s = -\frac{\partial}{\partial\rho} N(s)/(1 + N'(s)) > 0 \quad \text{for} \quad \rho \leq \sigma \, .$$

*Step 3: Large invariant intervals and the resulting bifurcation curve $\mathcal{C}_5$.* Now we pass to the special case $\sigma + \rho \leq 0$, $u_L \geq -q$, and consider equation

$$\zeta(\sigma, \rho) = 1 + 2e^{\sigma\pi} - s(\sigma, \rho) = 0 \, .$$

Since $A(-1) = L(-1) = 1 + 2e^{\sigma\pi} > 1 > u_R$, the monotonicity properties of function $A$ established in Theorem 2 show that $\mathbb{J} \subset \mathbb{R}$ is a large invariant interval $\Leftrightarrow \mathbb{J} = [-s, s]$ and $\zeta(\sigma, \rho) \leq 0$.

By using inequality (18), $\frac{\partial}{\partial\sigma}\zeta(\sigma, \rho) > 0$. The last result in Step 2 leads to $\frac{\partial}{\partial\rho}\zeta(\sigma, \rho) < 0$ (because inequality $\rho \leq \sigma$ is implied by $\sigma + \rho \leq 0$, $\sigma > 0$). On the other hand,

$$\zeta(0^+, \rho) = 2 + \frac{1}{\cos(\alpha)} \leq 0 \;\Leftrightarrow\; -\frac{\sqrt{3}}{3} \leq \rho < 0$$

whenever $\rho < 0$.

Now we turn our attention to the case $\sigma + \rho > 0$ and consider equation

$$\xi(\sigma, \rho) = u_R(\sigma, \rho) - s(\sigma, \rho) = 0 \, .$$

Arguing as before, we see that $\mathbb{J} \subset \mathbb{R}$ is a large invariant interval $\Leftrightarrow \mathbb{J} = [-s, s]$ and $\xi(\sigma, \rho) \leq 0$. Combining the major inequalities (17) and (18) of the previous two steps, $\frac{\partial}{\partial\sigma}\xi(\sigma, \rho) > 0$. But $\xi(0^+, \rho) = -\infty$ for $\rho > 0$ and $\xi(\infty, \rho) = \infty$ for $\rho \in \mathbb{R}$. Since

$$\frac{\partial}{\partial\rho} u_R(\sigma, \rho) = -\frac{2e^{\sigma(\pi+\beta)}\sin(\beta)}{v^2(\beta)} < 0 \, ,$$

we have also that $\frac{\partial}{\partial\rho}\xi(\sigma, \rho) < 0$ whenever $\rho \leq \sigma$.

Now existence and the desired properties of the function $\lambda_2$ follow from two separate applications of the implicit function theorem. The continuity of $\lambda_2$ at the separation point $\rho_* < 0$ (at the uniquely defined point $\rho_*$ separating the cases $\lambda_2(\rho) + \rho < 0$ and $\lambda_2(\rho) + \rho > 0$) follows from the limiting relation $v(\beta) \to 1$ as $\sigma + \rho \to 0$. Note also that $\frac{\partial}{\partial\rho}\lambda_2(\rho) > 0$ whenever $\rho \leq \sigma$ and $\rho \neq \rho_*$. (Computer simulations suggest that the last inequality holds true without the assumption $\rho \leq \sigma$.)

Finally, we take $\mathcal{C}_5 = \text{Graph}(\lambda_2)$. □

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