Cost Optimisation of Electric Power Transmission Networks Using Steiner Tree Theory

Kevin Prendergast and Doreen Thomas

***Abstract*— This paper introduces a new approach to electricity transmission network planning, which optimises the network with respect to capital cost. The approach is limited to three terminals at this stage, but it does illustrate the basic building blocks of a cost optimised network of larger size. Optimisation is achieved by means of a weighted Steiner tree method, in which the weight of a transmission line is the per unit length cost function. The angle between two connected lines is found, such that for a smaller angle a star network connecting the three terminals is more cost effective.**

I. INTRODUCTION

Environmental constraints and difficult approvals processes have limited expansion of transmission networks over the past few decades, and networks are being driven increasingly hard by means of operational strategies such as dynamic rating, but there is little capacity left to be captured. Yet load continues to grow, which will require new power stations, whose output must be transmitted to loads. Overlaying this dilemma is the intention of most governments to ensure considerably more generation is sourced from renewable resources. This will require a multitude of wind turbines, small hydro generators and other renewables, most of which will be remote from load centres and existing large power stations, and even if they happen to be close to the transmission network they will not be able to utilise it, because it cannot securely be driven any harder. In the USA a study has shown that in order to allow for the integration of 400 gigawatts of power from wind and other sources into the USA system, and to enable the power from the often remote resource centres to be moved to load centres, 19,000 miles of 765 kilovolt transmission line at a cost of approximately US$60 billion will be required. In Australia there are projections for 10 to 12 gigawatts of electrical power from wind by the year 2020, and at present there is no network to transmit that power to loads. The new power plants will be small by previous standards, for example 200 rather than 2000 megawatts (MW), and this will require many more transmission lines for interconnection. In many countries there is likely to be the need for new transmission networks overlaying the existing ones, connecting the multitude of small power stations to load centres.

Since there is very good information about location and intensity of wind, it is possible to determine location and size of future wind power plants. Therefore the opportunity exists, and, given the cost the motivation is real, to prepare conceptual designs of least cost transmission network overlays. Steiner tree theory does enable networks to be optimised with respect to length, and researchers at the University of Melbourne have done work for the Australian mining industry on the design of underground mines [1]. The total length of the transmission network is clearly a major determinant of cost. However length is not the only determinant; the rating of a transmission line, which is the maximum current the line may carry, also has a bearing on cost. Therefore a weighted cost approach is required. This has been done for a communication network [4], but that

Kevin Prendergast is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Victoria 3010, Australia 7500 kevinprendergast@bigpond.com

Doreen Thomas is Head of the Department of Mechanical Engineering, The University of Melbourne, Victoria 3010, Australia doreen.thomas@unimelb.edu.au

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approach is inadequate for an electrical power transmission network. This paper introduces an approach for minimising the cost of a transmission network, based on Steiner tree theory. It is necessarily limited in scope here, since it considers only three terminals, and is indicative only, but it has the potential to be developed into a useful tool in power system planning.

II. FUNDAMENTALS

A transmission line is a line of towers on which conductors are strung from end to end. At each end of a transmission line is switching plant. An electric power transmission network (network) is a network of transmission lines connecting points; these points are power stations, load substations, and sometimes switching substations. A transmission line can be single circuit or double circuit, the former consisting of a single path between its ends and the latter, two parallel and replicated paths on the one set of towers. Single circuit can be used when security, reliability, commercial and legal requirements are not violated by an outage between the ends. Double circuit can be used when such requirements would be violated by an outage of a single circuit line, though in some circumstances even a double circuit line might be deemed insufficient. In this paper cost optimisation is considered for networks of single circuit lines and networks of double circuit lines, and it is assumed that double circuit lines would meet all necessary requirements if single circuit did not. This assumption would normally be valid for generation from renewable resources because of the amount of the generation and its variability. The theory is developed for single circuit line networks, but is valid for double circuit.

Given a set $A = \{1, 2, ...n\}$ of n points in the Euclidean plane, we denote the transmission line connecting points $i, j \epsilon A$ by ij, and we let L be a *network of transmission lines* connecting the given points. Let E_{ij} be the set of *line end plant* of line $ij \epsilon L$, and let $E = \{E_{ij} : ij \epsilon L\}$. The transmission network N is $L \cup E$. We denote the cost of the transmission lines as C_L, that of the line end plant as C_E, and that of the network as C_T. The total cost C_T associated with a transmission line ij can be expressed as follows:

$$C_T(ij) = C_E(ij) + C_L(ij) \ .$$

A transmission line has a maximum current it may carry, because current in excess of this value could result in conductor sag that causes substandard ground clearance. The absolute value of this maximum current is known as the *rating* of the line. The line is designed for that current. Let I_{ij} be the current flowing in transmission line ij and $\hat{I}_{ij}$ be its rating. Then the above costs are as follows:

$$C_E(ij) = k_1 \ , \quad C_L(ij) = (k_2 + k_3 \hat{I}_{ij}) l_{ij} \ ,$$

where k_1, k_2, k_3 are constants, and l_{ij} is the Euclidean length of line ij.

In this paper we will consider the simple case of a network of single circuit (one circuit per line) 220 kV (220,000 volts) transmission lines. Indicative values for planning purposes of k_1, k_2, k_3 are given below. Easement acquisition is not included.

Remark 2.1: For a single circuit 220 kV transmission line, the cost function in dollars (Australian) per kilometre for comparison purposes may be taken as $C_L = k_2 + k_3 \hat{I} = 221,000 + 128 \hat{I}$.

Remark 2.2: The total cost for comparison purposes of terminal plant at both ends of a 220 kV transmission line may be taken as $k_1 = 3.6$ million Australian dollars.

Ignoring local considerations, k_1, k_2, k_3 are constant across the network. The total cost of the network N therefore is

$$C_T(N) = \sum_{ij \epsilon L} k_1 + \sum_{ij \epsilon L} (k_2 + k_3 \hat{I}_{ij}) l_{ij} \ ,$$

or

$$C_T(N) = C_E(N) + C_L(N) \ .$$

The cost $C_L(N)$ is analogous to the cost $C(N)$ in [4], which concerns flow-dependent communication networks. In the electrical transmission case rating replaces flow. The function $f(\hat{I}) = k_2 + k_3 \hat{I}$ is non-negative ($f(0) \geq 0$ and

$f(\hat{I}) > 0$ if $\hat{I} > 0$), non-decreasing ($f(\hat{I} + \hat{J}) \geq f(\hat{I})$ for any $\hat{J} > 0$), and satisfies the triangular inequality ($f(\hat{I}+\hat{J}) \leq f(\hat{I}) + f(\hat{J})$ for any $\hat{I}, \hat{J} > 0$). Therefore, according to [4], the network of transmission lines L is a *Gilbert* network. We call this function f the *weight* of the line, and denote it by the symbol w_{ij}.

We now introduce some Steiner theory concepts. A *cycle* is a closed loop path of lines. A *tree* is a network without cycles. In our three terminal case a tree can consist of two lines connecting the three terminals, and this is called a *spanning tree*. The minimum cost spanning tree is called the *Minimum Spanning Tree* (MST). For some configurations of terminals the least cost network is obtained by introducing a point that is not a terminal, and having this added point or junction connected to the terminals, in place of connections from terminal to terminal. Note that there is no generation or load at this point. The point is called the *Steiner point*, and the resultant star-shaped tree is know as the *Steiner Minimum Tree* (SMT). The location of the Steiner point depends on the metric, which is determined by the application. In the classical Euclidean case, where the Euclidean metric applies, the location of the Steiner point is well known [2], with the angles at the Steiner point being $\frac{2\pi}{3}$. A useful tool in Steiner theory is the variational technique [3], which uses the directional derivative. Suppose $e = ij$ is a line in L and j is perturbed to j' in a particular direction. Let $\dot{e}$ be the directional derivative of e in the direction of perturbation. Then $\dot{e} = -\cos(\angle ijj')$.

III. Minimum Cost Three Terminal Network

We take the transmission network to have a single nominal voltage. Then in this exercise we need only consider current, and for a given nominal voltage it is current that will determine the cost of the network. In a three terminal network there will be two generators and one load, or one generator and two loads. Networks of the physical dimensions and low frequency such as we are considering, are usually considered to have zero propagation time: a current appears at every point in a circuit simultaneously. This makes it impossible to speak of direction of current flow. However power flows from generator to load, and we adopt the convention that direction of current flow is from generator to load. This, via the following lemma, enables us to reduce the number of cases to consider.

Lemma 3.1: Consider a three terminal network $N = L \cup E$ with generators at terminals 1 and 2 having current outputs I_1 and I_2 respectively, and a load at terminal 3 having current input $I_3 = I_1 + I_2$. The cost of the network is unchanged when the generators at $1, 2$ are replaced with loads having input currents I_1, I_2 respectively, and the load at terminal 3 is replaced with a generator having current input $I_3 = I_1 + I_2$.

Proof. Let the current in line ij be I_{ij} for all lines ij in the network which has the generators at terminals $1, 2$ with current output I_1, I_2 respectively. Suppose the current in all lines is reversed such that there is a new set of currents $\{J_{ij} : ij \epsilon L\}$ with $J_{ji} = I_{ij}$. Then there are loads at terminals $1, 2$ with current input I_1, I_2 respectively, a generator at 3 as required, and $\hat{J_{ij}} = \hat{I_{ij}}$. Therefore, by the formula for C_T, the costs of the network to serve both conditions are equal. ∎

By Lemma 3.1 we only need to consider the case of generators at two terminals and load at one. When there is a cycle in a network we need to determine how the current divides in parallel paths in order to obtain line ratings. The following lemma states that, to an accuracy sufficient for planning purposes, line length determines the division of current. The following remarks provide essential data for Lemma 3.2.

Remark 3.1: A transmission line ij has impedance per km of $z_{ij} = r_{ij} + ix_{ij}$ ohm, where r_{ij} is the resistance, x_{ij} is the reactance and $i = \sqrt{-1}$. For transmission lines designed to transmit power from renewable sources into the power system, it is reasonable to assume the resistance will range from $r_{min} = 0.05$ to $r_{max} = 0.15$ ohm per km, and the reactance x will not vary significantly from 0.40 ohm per km.

Remark 3.2: Currents injected at the end points of a line will not vary significantly in phase angle.

Lemma 3.2: Suppose a delta connection $\triangle$ consisting of three lines $12, 23, 31$ occurs in a network, with injection of current I_1 at 1 and I_2 at 2. Then the currents I_{13}, I_{23}, I_{12} are approximately given by

$$I_{13} = \frac{l_{12} + l_{23}}{l_\triangle} I_1 + \frac{l_{23}}{l_\triangle} I_2$$

$$I_{23} = \frac{l_{13}}{l_\triangle} I_1 + \frac{l_{12} + l_{13}}{l_\triangle} I_2$$

$$I_{12} = \frac{l_{13}}{l_\triangle} I_1 - \frac{l_{23}}{l_\triangle} I_2 \ ,$$

where $l_\triangle = l_{12} + l_{23} + l_{31}$.

Proof. Application of Ohms Law gives

$$I_{13} = \frac{z_{12} + z_{23}}{z_\triangle} I_1 + \frac{z_{23}}{z_\triangle} I_2$$

$$I_{23} = \frac{z_{13}}{z_\triangle} I_1 + \frac{z_{12} + z_{13}}{z_\triangle} I_2$$

$$I_{12} = \frac{z_{13}}{z_\triangle} I_1 - \frac{z_{23}}{z_\triangle} I_2 \ ,$$

where $z_\triangle = z_{12} + z_{23} + z_{31}$.

For the reactance and maximum and minimum values of resistance in Remark 3.1, it can be shown that the absolute value of the ratios $\frac{z_{ij}+z_{jk}}{z_\triangle} / \frac{l_{ij}+l_{jk}}{l_\triangle}$, $i, j, k \epsilon \{1, 2, 3\}$, and $\frac{z_{ij}}{z_\triangle} / \frac{l_{ij}}{l_\triangle}$, $i \epsilon \{1, 2\}, j = 3$, lie between 0.94 and 1.06, with the extreme values occurring for triangle 123 having zero area. From Remark 3.2 the currents are approximately in phase, hence we conclude that line currents can be approximately determined from line lengths. ∎

Lemma 3.3: A network N consisting of two lines, 13 and 23, with current injection at terminals $1, 2$, has lower cost than the delta network $M = \{12, 23, 31\}$ with the same current injection.

Proof.

$$C_T(M) - C_T(N) = 3k_1 + k_2 l_\triangle + k_3(\hat{I_{12}} l_{12} + \hat{I_{23}} l_{23} + \hat{I_{13}} l_{13})$$

$$-2k_1 - k_2(l_{13} + l_{23}) - k_3(\hat{I}_1 l_{13} + \hat{I}_2 l_{23}) \ .$$

Without loss of generality we can assume $\hat{I}_1 l_{13} \geq \hat{I}_2 l_{23}$, whence $\hat{I_{12}} = \hat{I}_1 l_{13} / l_\triangle$. Therefore

$$C_T(M) - C_T(N) \geq k_1 + k_2 l_{12} + \frac{k_3}{l_\triangle} \hat{I}_2 l_{12} l_{23} \geq 0 \ .$$

∎

Lemma 3.4: A network consisting of two lines 13 and 12, with current injection at terminals $1, 2$, has lower cost than the delta network $\{12, 23, 31\}$ with the same current injection.

Proof. The reasoning is the same as in Lemma 3.3. ∎

So far, we do not know which of the networks $\{13, 23\}$ and $\{13, 12\}$ has the lower cost. To find out, we begin by letting $k = k_2 / k_3 \hat{I}_2$. This is the ratio of the cost per kilometre associated with the towers to the cost per kilometre associated with the conductors.

Lemma 3.5: Let $N = \{13, 23\}$ and $M = \{12, 13\}$, and suppose maximum currents $\hat{I}_1, \hat{I}_2$ are injected into each network at terminals $1, 2$ respectively. Then $C_T(N) \geq C_T(M)$ if and only if $l_{23} \geq l_{12} + \frac{1}{k+1} l_{13}$.

Proof. The cost difference between the networks may be written as

$$C_T(N) - C_T(M) = k_3 \hat{I}_2 (l_{23} - l_{12})(k - \frac{l_{12} + l_{13} - l_{23}}{l_{23} - l_{12}}) \ ,$$

and the result follows. ∎

If $\hat{I}_2 = 576$ ampere, which is equivalent to 165 MVA at 220 kV, then by Remark 2.1 we have $k = 3$, i.e. the cost associated with the towers is three times that associated with the conductors, and this would imply that the network $\{12, 13\}$ has the lower cost provided $l_{23} \geq l_{12} + \frac{1}{4} l_{13}$.

Lemma 3.6: For an SMT star network $N = \{1s, 2s, 3s\}$, where s is the Steiner point, with maximum current injections $\hat{I}_1, \hat{I}_2$ at terminals $1, 2$ respectively, the addition of a line $1'3'$ where $1', 3'$ are any points on $1s, 3s$ respectively, results in a network of higher cost. Similarly the addition of a line $1'2'$ also results in a network of higher cost.

Proof. By Lemma 3.3 the augmented network $\{1s, 2s, 3s, 1'3'\}$ has a higher cost than network $\{11'3', 2s3', 33'\}$, which in turn has a higher cost than network $\{13', 23', 33'\}$. By definition of SMT, network $\{13', 23', 33'\}$ has a higher cost than N. The augmented network $\{1s, 2s, 3s, 1'2'\}$ by Lemma 3.3 clearly has a higher cost than network $\{11's, 22's, 3s\} = N$. ∎

Lemma 3.7: For a two-line network $N = \{13, 23\}$ with maximum current injections $\hat{I}_1, \hat{I}_2$ at terminals $1, 2$ respectively, the addition of a line $1'2'$ where $1', 2'$ are any points on $13, 23$ respectively, results in a network of higher cost.
Proof. This follows from an application of Lemma 3.3. ∎

Theorem 3.8: For a three-terminal configuration with maximum current injections $\hat{I}_1, \hat{I}_2$ at terminals $1, 2$ respectively, the least cost network is either a two-line network or a star network.
Proof. This follows from Lemmas 3.6 and 3.7. ∎

For a sufficiently small angle between two lines ij, kj, a Steiner point s exists between ij, kj, such that the star network $\{is, js, ks\}$ is the least cost network connecting i, j, k. We need to find the maximum angle such that the SMT is a star network rather than a two-line network. This requires that we find the angle at the Steiner point between the lines connected to the generators, assuming the Steiner point has not degenerated into the apex of the triangle of terminals. First it is necessary to establish that if there is a Steiner point it is unique; Lemma 3.9 and Theorem 3.10 do this.

Lemma 3.9: The cost function is convex.
Proof. Each of the line length functions is convex, and $k_1, k_2, k_3 . \hat{I}_1 . \hat{I}_2$ are constant. Therefore the cost function is a linear combination of convex functions, which is also convex. ∎

Theorem 3.10: In an SMT $N = \{1s, 2s, 3s\}$ in which maximum currents $\hat{I}_1, \hat{I}_2$ can be injected at terminals $1, 2$ respectively, the angles $\angle(1s2), \angle(1s3)$ are given by

$$\angle(1s2) = \arccos \frac{-k_2^2 + 2k_3^2 \hat{I}_1 \hat{I}_2}{2(k_2 + k_3 \hat{I}_1)(k_2 + k_3 \hat{I}_2)} \ ,$$

$$\angle(1s3) = \arccos \frac{-k_2^2 - 4k_2 k_3 \hat{I}_1 - 2k_3^2 \hat{I}_1 - 2k_3^2 \hat{I}_1 \hat{I}_2}{2(k_2 + k_3 \hat{I}_1)(k_2 + k_3(\hat{I}_1 + \hat{I}_2))} \ .$$

Proof. Lemma 3.9 ensures that if there is a local Steiner point it is global. The local Steiner point, and hence the required angles, is found by the variational technique. Suppose the point s is perturbed to s' in the direction $\overrightarrow{3s}$. Let $\angle(1s2) = \theta$ and $\angle(1ss') = \theta_1$. Further, let $k_2 + k_3 \hat{I}_i = w_i$ for $i = 1, 2, 3$. Equating the directional derivative of the cost to zero we obtain

$$w_3 - w_1 \cos\theta_1 - w_2 \cos(\theta - \theta_1) = 0 \ .$$

Suppose the point s is perturbed in the direction normal to $\overrightarrow{3s}$. Equating this directional derivative of cost to zero yields

$$w_1 \sin\theta_1 - w_2 \sin(\theta - \theta_1) = 0 \ .$$

Eliminating θ_1 in the above two equations gives the required expression for $\angle(1s2)$. Perturbations of s in the direction of and normal to $\overrightarrow{2s}$ similarly yield the expression for $\angle(1s3)$. ∎

With the Euclidean metric, the three angles at the Steiner point of a star network would be 120 degrees. By comparison, for a least cost star network $N = \{1s, 2s, 3s\}$ with maximum current injection of 575 ampere at each of terminals 1 and 2, $\angle(1s2)$ is 102 degrees, and $\angle(1s3)$ and $\angle(2s3)$ are 129 degrees.

Theorem 3.11: The maximum angle between lines of a two-line network $M = \{13, 23\}$, such that a star network $N = \{1s, 2s, 3s\}$ is of lower cost, is obtained from the solution of the following equations:

$$s_x^2 + 2s_y^2 + (l_{12} - s_x)^2$$
$$= l_{12}^2 + 2\sqrt{s_x^2 + s_y^2}\sqrt{(l_{12} - s_x)^2 + s_y^2}\cos\theta \qquad (i)$$

$$s_x^2 + s_y^2 + (x - s_x)^2 + (y - s_y)^2$$
$$= x^2 + y^2 + 2\sqrt{s_x^2 + s_y^2}\sqrt{(x - s_x)^2 + (y - s_y)^2}\cos\alpha \quad (ii)$$

$$\frac{w_2 \sin\theta}{w_1 + w_2 \cos\theta} = \frac{s_x y - s_y x}{s_x(x - s_x) - s_y(y - s_y)} \qquad (iii)$$

$$k_1 + k_2(l_{1s} + l_{2s} + l_{3s}) + k_3(\hat{I}_1 l_{1s} + \hat{I}_2 l_{2s} + (\hat{I}_1 + \hat{I}_2) l_{3s})$$

$$= k_2(l_{13} + l_{23}) + k_3(\hat{I}_1 l_{13} + \hat{I}_2 l_{23}) \qquad (iv),$$

where $1 = (0,0)$, $2 = (l_{12}, 0)$, $3 = (x, y)$, $s = (s_x, s_y)$, $\theta = \angle(1s2)$, $\alpha = \angle(1s3)$, and l_{12} is given.

Proof. Without loss of generality, we may assume terminal 1 is at the origin and terminal 2 is on the x-axis and fixed. Equations *(i)* and *(ii)* are the cosine rule applied to triangles $1s2$ and $1s3$. Theorem 3.10 gives us angles θ and α, and with these applied in equations *(i)* and *(ii)* we obtain s_x and s_y as functions of x, y. The equation $w_1 \sin\theta_1 - w_2 \sin(\theta - \theta_1) = 0$ in Theorem 3.10 gives us θ_1, which describes the direction of line $3s$, and relating this to the coordinates of s and terminal 3 yields equation *(iii)*. The first three equations yield y as a function of x, and we now have s_x, s_y, y as functions of x. Equation *(iv)* expresses the fact that the costs of networks M and N are equal, and substitution of the expressions for s_x, s_y, x, y into the expressions for the line lengths in equation *(iv)* leads to a solution for x, and hence for s_x, s_y, y, from which the angle $\angle(132)$ may be obtained. ∎

We can obtain a realistic example of Theorem 3.11 by considering the case of $l_{12} = 100$ km and $I_1 = I_2 = 576$ amp. This case has angle $\angle(132) = 61.8\,\text{deg}$, which is considerably less than the $120\,\text{deg}$ in the Euclidean case. Finally, the case of the lower cost two-line network being $\{12, 13\}$ rather than $\{13, 23\}$ can be dealt with similarly, with the only change from Theorem 3.11 being the substitution of l_{12} for l_{23} in equation *(iv)*.

IV. A Practical Example

In the island state of Tasmania in Australia almost all electricity is generated in hydro-electric power stations. Hydro power was developed in Tasmania over a period of seventy years in the twentieth century. Consequently the transmission system was developed over the same period. In the early 1960s a double circuit 220 kV transmission line was built from Liapootah Power Station to Chapel street Substation, a major load centre in Hobart. About 15 years later a double circuit 220 kV line was built from the new Gordon Power Station to Chapel Street Substation. Had it been possible to plan both lines together, and if the above theory was applied, a star network would have been designed to connect the three terminals. Ignoring all other considerations, the star network would have been more than AUD10m cheaper (12%), using the cost model for double circuit lines, than the two-line network.

V. Further Work

The next step in this study could be to consider four terminal networks. Steiner points of degree 4 are possible, as the following example shows. Suppose we have four terminals $1, 2, 3, 4$ in a square formation of side length 100 km. Further, suppose equal maximum currents of 576 ampere can be injected into terminals $1, 2$ and equal maximum loads of 576 ampere exist at terminals $3, 4$. It can be shown that the network $\{1s, 2s, 3s, 4s\}$ is the least cost network.

Of course network augmentations could have more than 4 terminals, and it is necessary to consider how they could be treated. The more terminals there are the more difficult it becomes to find the SMT, and some strategies are necessary in order that useful results might be found. Some of these have already been identified in the study of Euclidean SMTs, and it will be worthwhile to see whether these can be extended to transmission networks. One such is the concept of a *Steiner hull*, in which regions known to contain an SMT are identified [2]. Another is the *lune property*. A lune is a lens-like area bordered by two arcs, each centred on the end of a Steiner edge, with radius equal to the length of the edge. A lune cannot contain any vertex of the SMT other than the those at the intersection of the arcs [2].

VI. Other Considerations

It is necessary to point out that other considerations could come into play in the conceptual design of a transmission network augmentation. One is terrain; even at the conceptual stage some line routes might have to be ruled out because of construction or approvals difficulties. Another is cost of losses, which is the capitalised cost of I^2r losses on the transmission line over the life of the line. This depends on the load duration profile on the line and the assumed cost of replacement generation, as well as the resistance of the conductors. It is unlikely that cost of losses will call for larger conductors than the Steiner tree approach for networks associated with the addition of renewable generation, because of the variable level of generation. In any case the consideration of cost of losses is a refinement that can come after route selection.

VII. Conclusion

This paper has demonstrated the genesis of an approach to transmission line planning, which enables network augmentation catering for the connection of a number of renewable generation stations to be optimised for cost.

VIII. ACKNOWLEDGMENTS

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