Analysis of Complex Networks

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Abstract

Our paper analyzes some new lines to advance on quickly evolving concepts, the so-called Complex Networks, represented by graphs in general. It will be very necessary to analyze the mutual relationship between some different concepts and their corresponding measures, with very interesting applications, as the case may be of Symmetry or Entropy, and Clustering Coefficient, for example.

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1. Introduction to Networks

We need to analyze here some very interrelated concepts about graphs, such as their Symmetry/Asymmetry degrees, their Entropies, etc. It may be applied when we study the different types of Systems; in particular, on Complex Networks [4, 9, 22].

A *system* can be defined as *a set of components functioning together as a whole.* A systemic point of view allows us to isolate a part of the world, and so, we can focus on those aspects that interact more closely than others.

[1]Budapest, July 2010

Network Science is a new scientific field that analyzes the interconnection among diverse networks, as for instance between Physics, Engineering, Biology, Semantics, and so on. Among its developers, we may remember Duncan Watts [24] with the *Small-World Network,* and Albert-László Barabasi [3, 4], who developed the *Scale-Free Network.* In his work, A.- L. Barabási found that the websites that form the network (of the WWW) have certain interesting mathematical properties [4, 9].

Network Theory is a quickly expanding area of Network and Computer Sciences, and may also be considered a part of the Graph Theory [7, 8, 9, 17].

Complex Networks are everywhere. Many phenomena in nature can be modeled as a network [19, 20, 22, 23], such as

- Brain structures. The brain as a network of neurons (as nodes), connected by synapses (their edges).
- Social interactions or the World Wide Web (WWW). All such systems can be represented in terms of nodes and edges.
- On Internet, the nodes represent routers, and the edges are represented by wires, or physical connections between them.
- In transport networks, the nodes can represent the cities and the edges the roads that connect them. These edges can have weights.

These networks are not random. The topology of different networks are very close. They are rooted in the Power Law, with a scale free structure [3, 9, 12]. How can very different systems have the same under-

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lying topological features? Searching the hidden laws of these networks, modeling and characterizing them are the current lines of research [6, 9, 13-16, 21, 24, 26].

2. Entropy

Entropy is a key concept in current science. For instance, it is essential on Information Theory. Because it describes the amount of "disorder" and randomness, and also the information content of a system. It gives us how much randomness is present in a random event, or in a signal. And obviously, it may be usefully applied to Complex Networks. Because Graph theory [7, 8, 11, 17] has emerged as a primary tool for detecting numerous hidden structures in various information networks, including Internet graphs, Social Networks, Biological Networks, or more generally, any graph representing relations on massive data sets. The analysis of these structures is very useful for introducing concepts such as Graph Entropy, and Graph Symmetry. Understanding the Web´s topology will be essential for making information very accesible on the WWW [1].

We consider a functional on a graph, $G = (V, E)$, with P a probability distribution on its node set, V. The mathematical construct called as *Graph Entropy* will be denoted by GE. Such function is convex. It tends to $+\infty$ on the boundary of the non-negative orthant of R^n. And it tends monotonically to $-\infty$ along rays from the origin. So, such minimum is always achieved, and it will be finite.

The *Entropy* of a system represents the amount of uncertainty one observer has about the state of the system. The simplest example of a system will be a random variable, which can be shown by a node into the graph, the representation of the mutual relationship between the edges.

The *Information* measures the amount of correlation between two systems, and it reduces to a mere difference between entropies. So, the *Entropy of a Graph* is a measure of graph structure, or lack of it [16]. Therefore, it may be interpreted as the amount of information, or the degree of *surprise* communicated by a message. And, since the basic unit of information is the bit, Entropy may also be viewed as the number of bits of "randomness" in the graph, validating that *the higher the entropy, the more random the graph is.*

The entropy of the degree distribution provides an average measurement of the heterogeneity of the complex network. Also it may be related to their robustness, or resilience to attacks.

Solé et al. [22] suggested the use of the remaining degree distribution *(RDD)* to compute such H value. Recall that the *RDD* of a terminal node of an edge will be defined as the number of edges connected to that node not counting the original edge. For Complex Networks, their entropy will be defined as

$$H(\mathbf{q}) = -\sum q(i) \;\; \log \; q(i)$$

where $\mathbf{q}$ is the remaining degree, given by the vector

$$\mathbf{q} = (q(1), \; q(2), ..., \; q(n))$$

and $q(i)$ is defined by

$$\mathbf{q}(i) = \frac{(i\;+\;1)\;\;P_{i+1}}{\langle i \rangle}$$

being P_i and $\langle i \rangle$ the distribution and the average of degrees, respectively.

We recall that given a random variable, X, its *Shannon Entropy* is given by

$$H(X) = -\sum P(x) \;\; \log_2 P(x)$$

whereas the *Rényi Entropy of order* $\alpha \neq 1$ of such random variable will be

$$H_\alpha(X) = \frac{1}{1-\alpha} \log_2 \left(\sum P(x)^\alpha\right)$$

The Renyi Entropy of order α converges to the Shannon Entropy, when $\alpha \to 1$, i.e.

$$lim_{\alpha\to 1} \left\{ \frac{1}{1-\alpha} \log_2 \left(\sum P(x)^\alpha\right) \right\} = \\ = -\sum P(x) \;\; \log_2 P(x)$$

So,

$$lim_{\alpha\to 1} H_\alpha(X) = H(X)$$

Therefore, the Rényi Entropy may be considered as a generalization of the Shannon Entropy, or dually expressed, the Shannon Entropy will be a particular case of Rényi Entropy.

The structural information content will be the entropy of the underlying graph topology. The extrema values of Entropy for a Graph, or a Complex Network, is reached when $H_{\max} = 1,$ in presence of a uniform degree distribution. And dually, his value will be $H_{\min} = 0,$ whenever all nodes have the same degree.

A method for determining the entropy of graphs, and therefore, of Complex Networks, is possible, essentially due to [7]. In such procedure, we assign a probability value to each node. For these, we use an information functional which quantifies the structure. So, it allows us determining its entropy. Firstly, it will be convenient to introduce a new geometrical tool, the so-called *j-spheres* of a graph. Given an unlabeled and connected n-graph, G, and let v_i be one of its nodes. Then, the *j-sphere of* v_i is

$$S_j\left(v_i, G\right) = \left\{v \in V: \ d\left(v_i, \ v\right) = j, \ being \ j \geq 1\right\}$$

From these system of expanding-contracting circles and the cardinalities of their node set which each one of them contains, we introduce the *information functional,* denoted as $f^V,$ by

$$f^V\left(v_i\right) = \exp_\beta\left\{\sum k_i \ card[S_i\left(v_i, G\right)]\right\}$$

where

$$k_\tau > 0, \ 1 \leq \tau \leq n, \ \beta > 0$$

Its signification is that it shows the structural information, being the coefficients, β and k_τ, useful weighting the different characteristics of the graph on each j-sphere.

The *probability value associated to each node* is given by

$$P^V\left(v_i\right) = \frac{f^V(v_i)}{\sum_{j\ =1}^{card(V)} f^V(v_j)}$$

And so, the *Entropy of G* would be expressable as

$$H_f^V\left(G\right) = -\sum_{i=1}^{card(V)} P^V\left(v_i\right) \ \log P^V\left(v_i\right)$$

We can consider other way [18] to access to the concept and computation of *Entropy,* as the *information content of a n-graph, G.* Departing to a decomposition of the set of nodes of G, or instead of the set of their edges, it is possible to construct a *finite probability scheme* (FPS), and then, compute its entropy. Recall that a *FPS* assigns to each subset in the above mentioned decomposition a probability value. So, this function gives us a measure of the information content of the graph.

As we known, the set of orbits of the *Aut (G)* -or automorphism group of a graph, *G*- constitute a decomposition of the set of the nodes *(V)* that belongs to the graph *G.* Such *Aut(G)* is composed by the set of all adjacency-preserving bijections of *V.*

Let $\{O_i\}_{i=1}^k$ be the set of their orbits, with $card\left(O_i\right) = n_i, \ 1 \leq i \leq k.$ Then, the *Entropy of* G may be defined as

$$H\left(G\right) = -\sum_{i=1}^{k} \frac{n_i}{n} \ \log \ \frac{n_i}{n}$$

It is clear that $0 \leq H\left(G\right) \leq \log \ n,$ reaching the minimum value when the graph possess a transitive automorphism group. An example may be the cycle C_n, or the complete n-graph, K_n.

3. Symmetry

As we known, *Symmetry* in a system means invariance of its elements under conditions of transformations [1, 7, 10, 16, 17]. By *Network,* we means a set of objects (nodes), connected by relations (edges). The *degree of a node* will be the number of edges connected to it. And the *degree distribution, P(k),* will be the probability that a randomly chosen node has degree *k.* When we take network structures, its *Symmetry* means *invariance of adjacency of nodes under the permutations on node set.* The graph isomorphism is an equivalence, as relation on the set of graphs. Therefore, it partitions the class of all graphs into equivalence classes. The underlying idea of isomorphism is that some objects have the same structure, if we omit the individual character of their components. A set of graphs isomorphic to each other is denominated as an *isomorphism class of graphs.*

The *automorphism of a graph,* $G = (V, E)$, will be an isomorphism from G onto itself. The family of all automorphisms of a graph G is a permutation group on *V(G).* The inner operation of such a group will be the composition of permutations. Its name is very well-known, as the *Automorphism Group of G,* and abridgedly, it is usually denoted by $Aut(G)$. And conversely, *all groups may be represented as the automorphism group of a connected graph.*

The *automorphism group* is an *algebraic invariant* of a graph. So, we can say that the *automorphism of a graph is a form of symmetry* in which the graph is mapped onto itself while preserving the edge-node connectivity. Such an automorphic tool may be applied both on *Directed Graphs* (DGs) and on *Undirected Graphs* (UGs).

Another interesting concept in Mathematics is the word *"genus",* which has different, but very related, meanings. So, in Topology, it depends on considering orientable or non-orientable surfaces. In the case of *connected and orientable surfaces,* it will be an integer that represents the maximum number of cuttings, along closed simple curves, without rendering the resultant manifold disconnected. For this reason, we may say that it is the number of *"handles"* on it. Usually, it is denoted by the letter g. It will be also definable by means of the *Euler number,* or *Euler Characteristic,* denoted by χ. Such a relationship will be expressed, for *closed surfaces,* by $\chi = 2-2g$. When the surface has b boundary components, this equation transforms into $\chi = 2 - 2g - b$, which obviously generalizes the above equation. For example, a *sphere,* an *annulus,* or a *disc* have genus $g = 0$. Instead of this, a *torus* has $g = 1$. In the case of *non-orientable surfaces,* the *genus* of a closed and connected surface will be a positive integer, representing the number of cross-caps attached to a sphere. Recall that a *cross-cap* is a two-dimensional surface that is topologically equivalent to a Möbius string. As in the precedent analysis, it can be expressed in terms of the Euler characteristic, by $\chi = 2 - 2k$, being k the *non-orientable genus.* For example, a *projective plane* has non-orientable genus $k = 1$. And a *Klein bottle* has a non-orientable genus $k = 2$.

Returning to a graph, its corresponding genus will be the minimal integer, n, so that the graph can be drawn without crossing itself on a sphere with *n handles.* So, a *planar graph* has genus $n = 0$, because it can be drawn on a sphere without self-crossing. In the *non-orientable case,* the genus will also be the minimal integer, n, so that the graph can be drawn without crossing itself on a sphere with *n cross-caps.* If we pass now to topological graph theory, we will define as *genus of a group, G,* the minimum genus of any of the undirected and connected Cayley graphs for G. From the viewpoint of the Computational Complexity, *the problem of "graph genus" is NP-complete.*

We will say either *graph invariant* or *graph property,* when it depends only on the abstract structure, not on graph representations, such as particular labelings or drawings of the graph [1]. So, we may define a *graph property* as every property that is preserved under all its possible isomorphisms of the graph. Therefore, it will be *a property of the graph itself,* not depending on the representation of the graph. The semantic difference also consists in its character: *qualitative or quantitative one.* For instance, when we said *"the graph does not possess directed edges",* this will be a *property,* because it is a *qualitative* statement. Whereas when we say *"the number of nodes of degree two in such graph",* this would be an *invariant,* because it is a *quantitative* statement.

From a strictly mathematical viewpoint, a *graph property* can be interpreted as *a class of graphs,* composed by the graphs that have in common the accomplishment of some conditions. Hence, a graph property can also be defined as a function whose domain would be the set of graphs, and its range will be the bivalued set composed by two options, true and false, according to which a determinate condition is either verified or violated for the graph [1, 7, 16, 17].

A graph property is called *hereditary*, if it is inherited by their induced subgraphs. And it is *additive,* if it is closed under disjoint union. For example, the property of a graph to be planar is both additive and hereditary. Instead of this, the property of being connected is neither. The computation of certain graph invariants may be very useful, with the purpose to discriminate when two graphs are isomorphic, or rather non-isomorphic. The support of these criteria will be that for any invariant at all, two graphs with different values cannot be isomorphic between them. But, nevertheless, two graphs with the same invariants may or may not be isomorphic between them. So, we get to the notion of *completeness.* A previous result said that *a digraph contains no cycle if and only if all eigenvalues of its adjacency matrix are equal to zero.* It is possible to prove that *every group is the automorphism group of a graph.* If the group is finite, the graph may be taken to be finite. And Pólya observed that *not every group is the automorphism group of a tree.*

The structural information content will be the entropy of the underlying graph topology. A method for determining the entropy of graphs, and therefore, of Complex networks, may be possible on each j-sphere. So, a network is said to be *asymmetric,* if its automorphism group reduces to the identity group. I.e. it only contains the identity permutation. Otherwise, the network is called *symmetric.* I.e. when the automorphism has elements different from to the identity [16, 17]. Current research has revealed a possibly surprising result, according to which the interaction networks displayed by most complex systems are highly heterogeneous.

If we consider the structure and size of *Aut (G),* for a real-world network, we can see that this will be greatly symmetric [18]. Because it is feasible to relate the automorphism group structure to the network topology, considering network symmetry via the automorphism group of the underlying graph. The basic step consists in the factorization of large network automorphism, which will be computationally efficient. Supposing G as a network, and being its associate group $Aut(G)$, if we denote by S the set of their generators, it will be partitionable into n minimal support-disjoint subsets, $\{S_i\}_{i=1}^{n}$.

Let H_i be the subgroup generated by S_i. Then,

$$Aut\,(G) \approxeq \prod H_i$$

and we can shown both unicity and irreducibility for this factorization. Because a certain degree of symmetry is very frequent. For this purpose, we will constructs a decomposition of the automorphism groups of these networks. Therefore, such groups can be decomposed into products of symmetric groups. So, each factor may be associated with a symmetric subgraph, and many factors can be related to a symmetric clique, or biclique. Such two types of subgraph account for almost all real-world network symmetry.

Also it is possible to introduce some new asymmetry and symmetry level measures, as by [16]. Let *(E, d)* be a fuzzy metric space. But our results are also generalizable to other spaces. We proceed to define the new fuzzy measure, which is a new and useful function. Such application would be defined as one of the kind L_i, with $i \in \{a,\ s\}$, where s denotes *symmetry*, and a denotes *asymmetry.* Suppose that from here we denote by $c\,(A)$ the cardinal of a fuzzy set A. We denote by $H[A]$ its entropy measure, and by $Sp\,(A)$ its corresponding specificity measure.

Theorem 1. Let *(E, d)* be a fuzzy metric space, being A a subset of *E,* and let H and Sp two fuzzy measures defined on *(E, d).* Then, the function L_s, operating on A as

$$L_s\,(A) = \left\{ Sp\,(A) \left(\frac{|1\ -\ c(A)|}{|1\ +\ c(A)|} \right) + (1 + H\,[A])^{-1} \right\}$$

will be also a fuzzy measure. This measure is called *Symmetry Level Function.*

Theorem 2. Let *(E, d)* be a fuzzy metric space, being A any subset of *E,* and let H and Sp two fuzzy measures defined on *(E, d).* Then, the function L_a, acting on A as

$$L_a\,(A) = 1 - \left\{ Sp\,(A) \left(\frac{|1\ -\ c(A)|}{|1\ +\ c(A)|} \right) + (1 + H\,[A])^{-1} \right\}$$

will be also a fuzzy measure. This measure is called *Asymmetry Level Function.*

Corollary 1. In the precedent hypothesis, the Symmetry Level Function is a Normal Fuzzy Measure.

Corollary 2. Also the Asymmetry Level Function is a Normal Fuzzy Measure.

Recall that the values of such fuzzy measure, *Sp*, decrease as the size of the considered set increases. Also that the Range of the Specificity Measure, *Sp*, will be the closed unit interval.

4. Complex Network Models

We may distinguish four structural models [3, 9, 12, 19, 22, 23, 26]. So, *Regular Networks, Random Networks, Small-World Networks,* and *Scale-free Networks.*

In the *Regular Network,* each node is connected to all other nodes. I.e. they are fully connected. Because of such a type of structure, they have the *lowest path length* (L), and the *lowest diameter* (D), being $L = D = 1$. Also, they have the *highest clustering coefficient.* So, it holds $C = 1$. Furthermore, the *highest possible number of edges*

$$card\ (E) = \frac{n\ (n-1)}{2} \sim n^2$$

As related to *Random Graphs (RGs),* we can say that each pair of nodes is connected with probability p. They have a *low average path length,* according to

$$L \approx \frac{\ln\ n}{\ln\ \langle k \rangle} \sim \ln\ n,\ \text{for}\ n >> 1$$

It is because the total network may be covered in $\langle k \rangle$ steps, from which

$$n \sim \langle k \rangle^L$$

Moreover, they possess a low clustering coefficient, when the graph is sparse. Thus,

$$C = p = \frac{\langle k \rangle}{n} << 1$$

given that the probability of each pair of neighboring nodes to be connected is precisely equal to p.

The *Small-World effect* is observed on a network when it has a low average path length, i.e.

$$L << n,\ for\ n >> 1$$

Recall the now famous *"six degrees of separation",* also called the *small-world phenomenon.* This hypothesis was first formulated by the Hungarian writer Frigyes Karinthy, in 1929. Then, it was tested by S. Milgram, in 1967. The subjacent idea is that two arbitrarily selected people may be connected by only six degrees of separation (in average, and it is not much larger than this value). Therefore, the diameter of the corresponding graph is not much larger than six. For instance, on social connections. So, the *Small-World property* will be interpreted as that despite its large size (of the corresponding graph), the shortest path between two nodes is small, as e.g. on WWW or Internet.

Self-similarity on network indicates that it is approximately similar to any part of itself, and therefore, it is fractal. In many cases, the real networks possess all these properties: they are Fractal, Small-World, and Scale-Free.

Fractal dimensions describe self-similarity of diverse phenomena, such as images, temporal signals, etc. Such fractal dimension gives us an indication of how completely a fractal appears to fill the space, as one zooms down to finer and finer scales. It is, so, a statistical quantity. The most important of such measures are *Rényi dimension, Hausdorff dimension,* and *Packing dimension.* Because their ease of implementation, the more usual procedures to compute such measures will be *Correlation dimension* and *Box counting.*

Fuzzy set approach may also produce more consistent models [16].

In the case of the *Watts-Strogatz* (by an acronym, WS) *Small-World model,* proposed in 1998, it represents a hybrid case between a Random Graph and a Regular Lattice [9, 24, 26]. So, Small-World models share with RGs some common features, such as the Poisson or Binomial degree distribution, near to Uniform type; network size: it does not grow; each node has approximately the same number of edges, i.e. it shows a homogeneous nature.

WS-models show the low average path length typical of Random Graphs,

$$L \sim \ln\ n,\ for\ n >> 1$$

And also such models give us the usual high clustering coefficient of Regular Lattices, being

$$C \approx 0.75,\ for\ k >> 1$$

In consequence, WS-models have a small-world structure, being well clustered. The Random Graphs coincide on the small-world structure, but they are poorly clustered. This model (WS) has a peak degree distribution, of Poisson type.

With reference to the last model [3, 9, 12, 16], called *Scale-Free Network,* this appears when the degree distribution follows a *Power-Law,* i.e.

$$P(k) \sim k^{-\gamma}$$

In such a case, there exist a small number of highly connected nodes, called *hubs*, which are the tail of the distribution. On the other hand, the great majority of the sets of their nodes have few connections, representing the head of such distribution.

Such a model was introduced by A.-L. Barabási and R. Albert, in 1999. Some of their *features* may be: *non-homogeneous nature,* in the sense that some (few) nodes have many edges from them, and the remaining nodes only have very few edges, or links; as related to the network size, *it continuously grows;* and regarding the connectivity, *it obeys a Power-Law distribution.* Many massive graphs, such as the WWW graph, share certain characteristics, described as such Power-Law.

Bollobás and Riordan [10] consider a Random Graph process in which nodes are added to the graph one at a time, and joined to a fixed number of earlier nodes, being each one of such earlier nodes with probability proportional to its degree. After n steps, the resulting graph have diameter approximatelely *log n*. This affirmation is true for $n = 1$. But for $n \geq 2$, the diameter value would be asymptotically

$$\frac{\log\ n}{\log(\log\ n)}$$

Another very interesting interesting mechanism is the so-called *Preferential Attachement process* (PA, in acronym). It will be any class of processes in which some quantity is distributed among a number of sets (for instance, objects or individuals), according to how much they already have, so that intuitively "the rich get richer", i.e. the more interrelated receive more new connections than those who are not. It will be interpreted as a current derivation of the Pareto Principle, very well-known on Economics. The principal scientific interest in PA is that it may to produce interesting power law distributions. Analytic solutions for PA mechanism were showed by Dogorotsev et al. (2000), and then, by Krapivsky et al., working on an independent way. But it was Bela Bollobás who proved this rigorously.

A very notable example of Scale-Free Network may be the *World Wide Web* (WWW, in acronym). As we know [1, 3, 14, 15], it is a collection of many possibly very different sub-networks. In such a case, the nodes correspond to their pages, whereas the edges correspond to hyperlinks. Related to the Web graph characteristics, we notice the *Scale Invariance* as being very important [21, 26]. Another interesting feature is the possibility to obtain the measurement of the World-Wide Web (its *diameter,* i.e. the shortest distance between any pair of nodes into the system), or at least a bound, either a mean value, or so on [1, 4, 25]. Because the WWW representation is made by a very large digraph, whose nodes are documents, and whose edges are links (URLs), pointing from one document to another.

Albert et al. found that the average of the shortest path between two nodes will be

$$\langle d \rangle = 0.35 + 2.06\ \log\ N$$

where N is the number of nodes in the Random Graph considered. This shown that *the Web is a SW network.* In particular, if we take $N = 8\ x\ 10^8$, we will obtain

$$\langle d_{Web} \rangle = 18.59$$

This result means that two randomly chosen nodes (documents), on the graph which represent the Web, are only on average nineteen steps, or clicks, from each other. For a given value of the number of nodes, N, the distribution associated to d is of Gaussian type. It will be also very remarkable the logarithmic dependence of such diameter on the value of *N.* In this sense, Albert et al. indicates that the future evaluation of $\langle d \rangle$, with the increasing of the Web, would change from 19 to only 21.

Conclusion

So, we have achieved our initial purpose, that of attempting to provide a comprehensive vision on the principal aspects and properties of Complex Networks, from a Mathematical Analysis point of view.

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