

# The Grothendieck problem with rank constraint

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**Abstract**—Finding a sparse/low rank solution in linear/semidefinite programming has many important applications, e.g. combinatorial optimization, compressed sensing, geometric embedding, sensor network localization. Here we consider one of the most basic problems involving semidefinite programs with rank constraints: the Grothendieck problem with rank- $k$ -constraint. It contains the MAX CUT problem as a special case when  $k = 1$ . We perform a complexity analysis of the problem by designing an approximation algorithm which is asymptotically optimal if one assumes the unique games conjecture.

## I. INTRODUCTION

Given positive integers  $m, n, k$  and a matrix  $A = (A_{ij}) \in \mathbb{R}^{m \times n}$ , the *Grothendieck problem with rank- $k$ -constraint* is defined as

$$\text{SDP}_k(A) = \max \left\{ \sum_{i=1}^m \sum_{j=1}^n A_{ij} x_i \cdot y_j : \right. \\ \left. x_1, \dots, x_m \in S^{k-1}, y_1, \dots, y_n \in S^{k-1} \right\},$$

where  $S^{k-1} = \{x \in \mathbb{R}^k : x \cdot x = 1\}$  is the unit sphere; the inner product matrix of the vectors  $x_1, \dots, x_m, y_1, \dots, y_n$  has rank  $k$ . This problem was introduced by Briët, Buhrman, and Toner [2] in the context of quantum nonlocality where they applied it to nonlocal XOR games.

The case  $k = 1$  is the classical Grothendieck problem where  $x_1, \dots, x_m, y_1, \dots, y_n \in \{-1, +1\}$ . It was introduced by Grothendieck [5] in the study of norms of tensor products of Banach spaces. It is an NP-hard problem: If  $A$  is the Laplacian matrix of a graph then  $\text{SDP}_1(A)$  coincides with the value of a maximum cut of the graph. The maximum cut problem (MAX CUT) is one of Karp's 21 NP-complete problems.

Over the last years, there has been a lot of work on algorithmic applications, interpretations and generalizations of

The first author is supported by Vici grant 639.023.302 from the Netherlands Organization for Scientific Research (NWO), by the European Commission under the Integrated Project Qubit Applications (QAP) funded by the IST directorate as Contract Number 015848, and by the Dutch BSIK/BRICKS project. The second author was partially supported by CAPES/Brazil under grant BEX 2421/04-6, and the research was carried out at the Centrum Wiskunde & Informatica, The Netherlands. The third author is supported by Vidi grant 639.032.917 from the Netherlands Organization for Scientific Research (NWO).

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the Grothendieck problem and the companion Grothendieck inequalities. For instance, Nesterov [10] showed that it has applications to finding and analyzing semidefinite relaxations of nonconvex quadratic optimization problems. Alon and Naor [1] showed that it has applications to constructing Szemerédi partitions of graphs and to estimating the cut norms of matrices. Khot and Naor [7], [8] showed that it has applications to kernel clustering.

When  $k$  is a constant that does not depend on the matrix size  $m, n$  there is no polynomial-time algorithm known which solves  $\text{SDP}_k$ . However, it is not known if the problem  $\text{SDP}_k$  is NP-hard when  $k \geq 2$ . On the other hand the *semidefinite relaxation* of  $\text{SDP}_k(A)$  defined by

$$\text{SDP}_\infty(A) = \max \left\{ \sum_{i=1}^m \sum_{j=1}^n A_{ij} u_i \cdot v_j : \right. \\ \left. u_1, \dots, u_m, v_1, \dots, v_n \in S^\infty \right\}$$

can be computed in polynomial time using semidefinite programming. Here  $S^\infty$  denotes the unit sphere of the Hilbert space  $l^2(\mathbb{R})$  of square summable sequences, which contains  $\mathbb{R}^n$  as the subspace of the first  $n$  components.

Rietz [12] (in the context of the Grothendieck inequality) showed that  $\text{SDP}_1$  and  $\text{SDP}_\infty$  are always within a factor of at most  $(4/\pi - 1)^{-1} = 3.65979\dots$  from each other. That is, for all matrices  $A \in \mathbb{R}^{m \times n}$  we have

$$\text{SDP}_1(A) \leq \text{SDP}_\infty(A) \leq \frac{1}{4/\pi - 1} \text{SDP}_1(A).$$

Alon and Naor [1] showed that Rietz' argument gives a polynomial-time approximation algorithm for  $\text{SDP}_1$ . However Rietz' argument does not provide the best approximation factor. The best approximation factor is called *Grothendieck's constant*  $K_G$  which presently is not known exactly. It is known that it lies between  $1.67695\dots$  and  $\pi/(2\log(1 + \sqrt{2})) = 1.78221\dots$ . Krivine [9] gave the best known upper bound (which is conjectured to be tight) and Davie [4] and Reeds [11] gave the best known lower bound.

The aim of this paper is to provide to initiate an analysis for  $\text{SDP}_k$  by generalizing Rietz' argument. So far the only cases which were studied are the cases  $k = 1$  (see the discussion above) and the case  $k = 2$  (see the discussion below). We summarize our results in the following theorem.

**Theorem.** *For all matrices  $A \in \mathbb{R}^{m \times n}$  we have*

$$\text{SDP}_k(A) \leq \text{SDP}_\infty(A) \leq \frac{1}{2\gamma(k) - 1} \text{SDP}_k(A),$$

where

$$\gamma(k) = \frac{2}{k} \left( \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} \right)^2 = 1 - \Theta(1/k),$$

and

$$\frac{1}{2\gamma(k) - 1} = 1 + \Theta(1/k),$$

and there is a randomized polynomial-time approximation algorithm for  $\text{SDP}_k$  achieving this ratio. On the other hand, under the assumption of the unique games conjecture there is no polynomial-time algorithm which approximates  $\text{SDP}_k$  with an approximation ratio less than  $1/\gamma(k)$ .

The first three values of  $\frac{1}{2\gamma(k)-1}$  are:

$$\begin{aligned} \frac{1}{2\gamma(1) - 1} &= \frac{1}{4/\pi - 1} = 3.65979\dots, \\ \frac{1}{2\gamma(2) - 1} &= \frac{1}{\pi/2 - 1} = 1.75193\dots, \\ \frac{1}{2\gamma(3) - 1} &= \frac{1}{16/(3\pi) - 1} = 1.43337\dots \end{aligned}$$

Haagerup [6] gave an approximation ratio of  $1.40491\dots$  for the case  $k = 2$  by following Krivine's proof.

In Section II we present the approximation algorithm together with its analysis. Our main contribution is the analysis of a rounding scheme which can deal with rank- $k$ -constraints in semidefinite programs. For this we use the Wishart distribution from multivariate statistics. We believe this analysis is of independent interest and will turn out to be useful in different contexts, e.g. for approximating low dimensional geometric embeddings.

The proof of the unique games conjecture hardness of approximating  $\text{SDP}_k$  can be found in Briët, Oliveira, Vallentin [3].

The main result of Briët, Buhrmann and Toner [2] shows that the gap between  $\text{SDP}_k$  and  $\text{SDP}_\infty$  given in our theorem is asymptotically correct up to lower order terms.

## II. APPROXIMATION ALGORITHM

The following proof follows the idea of Alon and Naor [1, Section 4] which in turn relies on ideas of Rietz [12]

*Proof:* The randomized polynomial-time approximation algorithm which we use to prove the theorem is the following three-step process.

- 1) By solving  $\text{SDP}_\infty(A)$  we obtain the vectors  $u_1, \dots, u_m, v_1, \dots, v_n \in S^{m+n-1}$ .
- 2) Choose  $Z = (Z_{ij}) \in \mathbb{R}^{k \times (m+n-1)}$  so that every matrix entry  $Z_{ij}$  is distributed independently according to the standard normal distribution with mean 0 and variance 1:  $Z_{ij} \sim N(0, 1)$ .
- 3) Set  $x_i = Zu_i / \|Zu_i\| \in S^{k-1}$  with  $i = 1, \dots, m$ , and  $y_j = Zv_j / \|Zv_j\| \in S^{k-1}$  with  $j = 1, \dots, n$ .

The quality of the feasible solution  $x_1, \dots, x_m, y_1, \dots, y_n$  for  $\text{SDP}_k$  is measured by the expectation

$$\text{SDP}_k(A) \geq \mathbb{E} \left[ \sum_{i=1}^m \sum_{j=1}^n A_{ij} x_i \cdot y_j \right].$$

For vectors  $u, v \in S^\infty$  we define

$$E_k(u, v) = \mathbb{E} \left[ \frac{Zu}{\|Zu\|} \cdot \frac{Zv}{\|Zv\|} \right],$$

where  $Z = (Z_{ij})$  is a matrix with  $k$  rows and infinitely many columns whose entries are distributed independently according to the standard normal distribution. Of course, if  $u, v \in S^{m+n-1}$ , then it suffices to work with finite matrices  $Z \in \mathbb{R}^{k \times (m+n-1)}$ . In Briët, Oliveira, Vallentin [3] it was shown that one can develop  $E_k$  as a power series

$$E_k(u, v) = \sum_{r=0}^{\infty} f_{2r+1}(u \cdot v)^{2r+1},$$

where all coefficients  $f_{2r+1}$  are nonnegative, where

$$\begin{aligned} f_1 &= \frac{\partial E_k}{\partial t}(0) \\ &= \frac{k-1}{2\pi} \int_0^1 \int_0^{2\pi} \frac{r(1-r^2)^{(k-1)/2}}{(1-r^2(\sin \phi)^2)^{3/2}} d\phi dr. \\ &= \gamma(k). \end{aligned}$$

and where

$$\sum_{r=0}^{\infty} f_{2r+1} = 1.$$

For the computation of  $f_1$  we used the Wishart distribution from multivariate statistics: Because  $E_k$  is invariant under the orthogonal group one can express the integral  $E_k(u, v)$  with help of the standard Wishart distribution  $W_2(k)$ . This is the probability distribution of random matrices  $Y = X^T X \in \mathbb{R}^{2 \times 2}$ , where the entries of the matrix  $X = (X_{ij}) \in \mathbb{R}^{k \times 2}$  are independently chosen from the standard normal distribution  $X_{ij} \sim N(0, 1)$ .

Now we have

$$\begin{aligned} &\mathbb{E} \left[ \sum_{i=1}^m \sum_{j=1}^n A_{ij} x_i \cdot y_j \right] \\ &= \sum_{i=1}^m \sum_{j=1}^n A_{ij} E_k(v_i, w_j) \\ &= f_1 \sum_{i=1}^m \sum_{j=1}^n A_{ij} v_i \cdot w_j \\ &\quad + \sum_{i=1}^m \sum_{j=1}^n A_{ij} \sum_{r=1}^{\infty} f_{2r+1}(v_i \cdot w_j)^{2r+1}. \end{aligned}$$

The first summand equals  $f_1 \text{SDP}_\infty(A)$ . The second summand is bounded in absolute value by  $(1 - f_1) \text{SDP}_\infty(A)$  as we argue now. Consider the  $m+n$  vectors  $u_i = v_i$ ,  $u_{m+i} = w_j$ . Then the  $(m+n) \times (m+n)$ -matrix with entries

$$\sum_{r=1}^{\infty} f_{2r+1}(u_i \cdot u_j)^{2r+1}$$

is positive semidefinite because of Schoenberg's theorem [13]. So there are vectors  $u'_1, \dots, u'_{m+n} \in \mathbb{R}^{m+n}$  all of squared length  $1 - f_1$  so that

$$u'_i \cdot u'_j = \sum_{r=1}^{\infty} f_{2r+1}(u_i \cdot u_j)^{2r+1}.$$

Setting  $v'_i = u_i$ ,  $w'_j = u_{m+j}$  we have

$$\sum_{i=1}^m \sum_{j=1}^n A_{ij} v'_i \cdot w'_j \leq (1 - f_1) \text{SDP}_\infty(A).$$

The lower bound follows from the fact that replacing the sign of  $v_i$  to  $-v_i$  also changes the sign of the complete sum from

$$\sum_{r=1}^{\infty} f_{2r+1} (v_i \cdot w_j)^{2r+1}$$

to

$$- \sum_{r=1}^{\infty} f_{2r+1} (v_i \cdot w_j)^{2r+1}$$

since all involved powers are odd. Thus for the second sum we have

$$\sum_{i=1}^m \sum_{j=1}^n A_{ij} \sum_{r=1}^{\infty} f_{2r+1} (v_i \cdot w_j)^{2r+1} \geq (f_1 - 1) \text{SDP}_\infty(A),$$

which finishes the proof. ■

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