The design and parametrization of stabilizing feedback compensators via injective cogenerator quotient signal modules

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Abstract— **The design and parametrization of stabilizing feedback compensators which realize various goals like tracking, disturbance rejection, decoupling, model matching and others belong to the most important and difficult tasks of control engineering and have therefore been treated by many prominent researchers and in many textbooks, the systems being generally described by their transfer matrices or by Rosenbrock equations. Our approach to these important problems uses, in addition to the ideas of our predecessors, a new mathematical technique and is distinguished by the following features:**

1) **The plant, the compensator and the full feedback system are given as input/output behaviors. We study the full feedback behavior and especially its autonomous part and not only its transfer matrix or an induced manifest controlled behavior (like Willems et al.).**
2) **We simultaneously treat continuous and discrete systems and different notions of stability, for instance asymptotic and dead-beat stability, which are defined by multiplicatively closed sets T of T-stable polynomials and their quotient rings of T-stable rational functions.**
3) **We solve the problem of pole placement or spectral assignability for the complete feedback behavior.**
4) **We use an injective cogenerator signal module $\mathcal{F}$ over the polynomial algebra like all standard signal spaces and its quotient module $\mathcal{F}_T$ which is an injective cogenerator over the ring of stable rational functions and induces a categorical duality between finitely generated modules over this ring and $\mathcal{F}_T$-behaviors which represent the essential part of the behaviors in stability problems. The new duality technique enables very short and conceptual proofs both of standard results and also of sharper new ones. For instance, under the assumption of T-stabilizability we parametrize all tracking stabilizing compensators of a *not necessarily proper* plant such that both the feedback behavior and the compensator itself are proper. The first property ensures the absence of impulsive solutions in the continuous case, and the second property enables the realization of the compensator by Kalman equations or elementary building blocks. We notice that every behavior admits an input/output decomposition with proper transfer matrix, but that most of these decompositions do not have this property, and therefore we do not assume the properness of the plant.**

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INTRODUCTION

Problems of control design have always been of central interest in systems theory and have been investigated by many prominent researchers, among them Antsaklis and Michel [1, Ch.7, Part 2, pp.589-634], Bengtsson, Callier and Desoer [4, Chs.7,9, pp.196-242], Chen [5, Ch.9, pp.458-534], Falb, Francis, Kailath [6, Sect.7.5, pp.532-538], Kučera, Murray, Pernebo [9], Saeks, Schneider, Vardulakis [11, Ch.7, pp.335-354], Vidyasagar [12, Sect.5.7, Sect.7.5, pp.294-317], Wolovich [13, Ch.8, pp.269-323], Wonham, Youla, Zames, their coauthors and many other contributors. We refer to the quoted books for history, origin and development of the decisive ideas of control design. In addition to the ideas of our predecessors our approach uses several new ingredients, the most significant of those being arbitrary monoids of stable polynomials and the induced injective cogenerator quotient signal modules from [8] and [2].

The systems in the literature are generally described by Rosenbrock equations or often by their transfer matrix only. In our approach the plant is an input/output (IO) behavior $\mathcal{B}_1$ which is stabilized via output feedback by a proper IO behavior $\mathcal{B}_2$, the compensator, such that the total feedback behavior $\mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is well-posed, stable and proper and performs desired tasks like tracking, disturbance rejection, model matching or decoupling. In particular, we study the autonomous part of the feedback behavior in detail. The attribute *proper* refers to the properness of the transfer matrix. In contrast to the literature we do not assume properness of the plant $\mathcal{B}_1$ and can therefore admit arbitrary decompositions of the variables of $\mathcal{B}_1$ into input and output variables. According to [8], [3] and [2] stability is defined with respect to a saturated multiplicatively closed set (monoid) T of polynomials, interpreted as differential or difference operators, and called T*-stability*. This permits to treat different forms of stability simultaneously such as, for instance, asymptotic stability or, in the discrete case, dead-beat stability. Properness of $\mathcal{B}_2$ is needed for its realization by Kalman state space equations and elementary building blocks. Properness of $\mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is required in order to prevent, in the continuous case, impulsive trajectories whose input components are bounded piecewise continuous functions, but whose output components contain Dirac impulses.

We show that the not necessarily proper plant $\mathcal{B}_1$ is T-stabilizable, i.e., admits a T-stabilizing compensator, if and

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only if the quotient behavior $\mathcal{B}_{1,T}$ is controllable. Then it also admits proper T-stabilizing compensators $\mathcal{B}_2$ with proper and T-stable feedback behavior $\mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$, and we parametrize the set of all these proper compensators constructively. The generality of the monoid T moreover permits to solve the problem of spectral assignability or pole placement: The smallest monoid T_1 for which $\mathcal{B}_1$ is T_1-stabilizable is finitely generated and can be easily determined. The finitely many roots of the polynomials in T_1 are then unavoidable as poles of the feedback behavior whereas the other poles can be chosen arbitrarily.

Among all T-stabilizing compensators $\mathcal{B}_2$ thus obtained we then constructively characterize those which perform the desired tasks like tracking etc., and for this restricted class we also solve the problems of constructive parametrization and of spectral assignability. Several new algorithms support the constructive solutions.

PRELIMINARIES

Let $\mathcal{D}$ denote the polynomial ring $F[s]$ over some field F with characteristic 0, $\mathcal{K} := \mathrm{quot}(\mathcal{D}) = F(s)$ its quotient field, and $\mathcal{F}$ an injective cogenerator over $\mathcal{D}$. The standard choices are the following: $F = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{F} = \mathcal{C}^\infty(\mathbb{R}, F)$ or $\mathcal{D}'(\mathbb{R}, F)$ (continuous standard case) or $\mathcal{F} = F^{\mathbb{N}}$ (discrete standard case) with $s \circ y := \frac{dy}{dt}$ resp. $(s \circ y)(t) := y(t+1)$ for $y \in \mathcal{F}$ in the continuous resp. discrete standard case. Furthermore, let T be a multiplicatively closed saturated subset of $\mathcal{D} \setminus \{0\}$ and $\mathcal{D}_T := \left\{\frac{d}{t} \in F(s);\, d \in \mathcal{D},\, t \in T\right\}$ the quotient ring of $\mathcal{D}$ w.r.t. T. We refer to polynomials $t \in T$ as *T-stable polynomials*, and to elements $\frac{d}{t} \in \mathcal{D}_T$ as *T-stable rational functions*. For any $\mathcal{D}$-module M, denote by $M_T := \left\{\frac{x}{t};\, x \in M,\, t \in T\right\}$ the quotient module of M w.r.t. T which is a $\mathcal{D}_T$-module in the natural way. The assignment $(-) \to (-)_T$ is an exact functor. The quotient $\mathcal{F}_T$ of the signal module is even an injective cogenerator over $\mathcal{D}_T$, cf. [2, Thm. 1.6]. For further properties of $\mathcal{F}_T$ and the quotient behavior $\mathcal{B}_T$ of a behavior $\mathcal{B} \subseteq \mathcal{F}^\ell$ see [2, Thm. 1.6-Rem. 1.11]. We call a behavior $\mathcal{B} = \left\{w \in \mathcal{F}^\ell;\, R \circ w = 0\right\}$ where $R \in \mathcal{D}^{k \times \ell}$ *T-autonomous* if there exists $t \in T$ such that $t \circ \mathcal{B} = 0$. This is equivalent to $\mathcal{B}_T = \left\{w \in \mathcal{F}_T^\ell;\, R \circ w = 0\right\} = 0$ or to the existence of a left inverse matrix of R in $\mathcal{D}_T^{\ell \times k}$. Signals which are annihilated by some $t \in T$ are called *T-small*. We call an input/output behavior $\mathcal{B} = \{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\, P \circ y = Q \circ u\}$, $(P,\, -Q) \in \mathcal{D}^{p \times (p+m)}$, $\det(P) \neq 0$, *T-stable* if its autonomous part $\mathcal{B}^0 := \{y \in \mathcal{F}^p;\, P \circ y = 0\}$ is T-autonomous, compare [3, Thm.+Def. 2.15].

The following lemma will be applied repeatedly for different rings in the sequel.

Lemma 0.1: Let R denote a principal ideal domain with quotient field K. Assume a matrix $H \in K^{p \times m}$.

1) There exists an essentially unique (i.e., unique up to row equivalence over R) matrix $(P,\, -Q) \in R^{p \times (p+m)}$ which satisfies the following equivalent conditions with $U := R^{1 \times p}(P,\, -Q)$:
   a) The sequence
   $$0 \longrightarrow R^{1 \times p} \xrightarrow{\circ(P,\, -Q)} R^{1 \times (p+m)} \xrightarrow{\circ\left(\begin{smallmatrix} H \\ \mathrm{id}_m \end{smallmatrix}\right)} K^{1 \times m}$$
   is exact.
   b) i) $PH = Q$, i.e., $(P,\, -Q)\left(\begin{smallmatrix} H \\ \mathrm{id}_m \end{smallmatrix}\right) = 0$, and
   ii) $(P,\, -Q)$ has a right inverse in $R^{(p+m) \times p}$, i.e., $\mathrm{rank}(P,\, -Q) = \dim_R(U) = p$ and U is a direct summand of $R^{1 \times (p+m)}$ or $\dim_R(U) = p$ and the elementary divisors of U (or $(P,\, -Q)$) are units in R.

   In this case $R^{1 \times p}P = \left\{\xi \in \mathcal{D}^{1 \times p};\, \xi H \in R^{1 \times m}\right\}$, $\det(P) \neq 0$, and $H = P^{-1}Q$. The representation $H = P^{-1}Q$ is called the *left coprime factorization* (l.c.f.) and $(P,\, -Q)$ the *controllable realization* of H over R.
2) Likewise, there is an essentially unique (i.e., unique up to column equivalence over R) matrix $\left(\begin{smallmatrix} N \\ D \end{smallmatrix}\right) \in R^{(p+m) \times m}$ such that $HD = N$ and $\left(\begin{smallmatrix} N \\ D \end{smallmatrix}\right)$ has a left inverse in $R^{m \times (p+m)}$, i.e.,
   $$0 \longrightarrow R^m \xrightarrow{\left(\begin{smallmatrix} N \\ D \end{smallmatrix}\right)\circ} R^{p+m} \xrightarrow{(\mathrm{id}_p,\, -H)\circ} K^p$$
   is exact. Then $\det(D) \neq 0$ and $H = ND^{-1}$ is called the *right coprime factorization* (r.c.f.) of H over R.

I. STABILIZATION VIA OUTPUT FEEDBACK

In the following, we consider two IO behaviors:

$$\mathcal{B}_1 = \left\{\left(\begin{smallmatrix} y_1 \\ u_1 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\, P_1 \circ y_1 = Q_1 \circ u_1\right\}, \qquad (1)$$
$$\mathcal{B}_2 = \left\{\left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\, P_2 \circ y_2 = Q_2 \circ u_2\right\} \qquad (2)$$

where $(P_1,\, -Q_1) \in \mathcal{D}^{p \times (p+m)}$, $\det(P_1) \neq 0$, $(-Q_2,\, P_2) \in \mathcal{D}^{m \times (p+m)}$, $\det(P_2) \neq 0$. Their associated modules of equations are

$$U_1 := \mathcal{D}^{1 \times p}(P_1,\, -Q_1), \quad U_2 := \mathcal{D}^{1 \times m}(-Q_2,\, P_2).$$

Definition 1.1: The *feedback behavior* is defined as

$$\mathcal{B} := \mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2) := \left\{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{(p+m)+(p+m)};\, P \circ y = Q \circ u\right\}$$
$$\text{where} \quad y := \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right),\ u := \left(\begin{smallmatrix} u_2 \\ u_1 \end{smallmatrix}\right),$$
$$P := \begin{pmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{pmatrix},\ Q := \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix}$$

with module of equations $U = \mathcal{D}^{1 \times (p+m)}(P,\, -Q)$. Then $U^0 := \mathcal{D}^{1 \times (p+m)}P = U_1 + U_2$. The feedback behavior is called *well-posed* if it is an IO behavior with input $u = \left(\begin{smallmatrix} u_2 \\ u_1 \end{smallmatrix}\right)$ or, equivalently, if $\mathcal{B}^0 := \{y \in \mathcal{F}^{p+m};\, P \circ y = 0\}$ is autonomous, i.e., $\det(P) \neq 0$. This signifies that

$$U^0 = U_1 + U_2 = U_1 \oplus U_2.$$

The feedback interconnection is displayed in Figure 1.

Theorem and Definition 1.2: For $\mathcal{B} = \mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$ the following conditions are equivalent:

1) The feedback behavior $\mathcal{B}$ is well-posed and T-stable, i.e., $\mathcal{B}^0$ is T-autonomous or $\mathcal{B}_T^0 = 0$.
2) The matrix P is invertible in $\mathcal{D}_T$, i.e., $\det(P) \in T$.

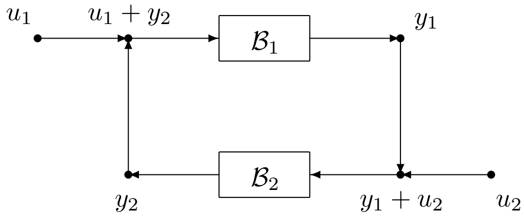

Fig. 1. The feedback behavior $\mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$.

3) a) The quotient behavior $\mathcal{B}_T$ is controllable and
   b) $\mathcal{B}$ is well-posed and the transfer matrix $H := P^{-1}Q$ is contained in $\mathcal{D}_T^{(p+m)\times(p+m)}$.

4) $U_{1,T} \oplus U_{2,T} = \mathcal{D}_T^{1\times(p+m)}$.

In this case $\mathcal{B}_2$ is called a *T-stabilizing compensator* of $\mathcal{B}_1$. The behavior $\mathcal{B}_1$ is said to be *T-stabilizable* if there exists a T-stabilizing compensator.

Note that T-stabilizability of a given IO behavior $\mathcal{B}_1$ implies that $U_{1,T}$ is a direct summand of $\mathcal{D}_T^{1\times(p+m)}$, i.e., that $(P_1, -Q_1)$ has a right inverse in $\mathcal{D}_T^{(p+m)\times p}$ or that all elementary divisors of $(P_1, -Q_1)$ (w.r.t. $\mathcal{D}_T$) are units in $\mathcal{D}_T$. This is equivalent to controllability of $\mathcal{B}_{1,T}$ (compare e.g. [10, 5.2.10][3, p. 2419][7, Thm. 7.21, 7.52, 7.53, pp. 141-152]).

It is possible to characterize T-stabilizability and to parametrize all T-stabilizing compensators $\mathcal{B}_2$ for given $\mathcal{B}_1$ on the basis of the previous result. However, our main interest is in *proper* compensators $\mathcal{B}_2$ such that also the feedback behavior is *proper* (i.e., with proper transfer matrix).

In order to study problems related to properness, we introduce the rings of proper resp. of proper and T-stable rational functions

$$F(s)_{\mathrm{pr}} := \left\{ \tfrac{f}{g} \in F(s);\ \deg\left(\tfrac{f}{g}\right) := \deg(f) - \deg(g) \le 0 \right\}$$
$$\text{resp.} \quad \mathcal{S} := \mathcal{D}_T \cap F(s)_{\mathrm{pr}}.$$

We will always assume that the set T contains an element $(s-\alpha)$ where $\alpha \in F$. Otherwise (in the case $F = \mathbb{C}$), if $T = \{1\}$ (or $T = F \setminus \{0\}$), $F(s)_{\mathrm{pr}} \cap \mathcal{D}_T = F$, i.e., the only proper T-stable rational functions are constant. With the notations

$$\sigma := \frac{1}{s-\alpha} \quad \text{and} \quad \widehat{\mathcal{D}} := F[\sigma],$$

both the rings $F(s)_{\mathrm{pr}}$ and $\mathcal{S}$ are quotient rings of $\widehat{\mathcal{D}}$, viz.

$$F(s)_{\mathrm{pr}} = \widehat{\mathcal{D}}_{\widehat{\mathcal{D}}\setminus\widehat{\mathcal{D}}\sigma} = \left\{ \tfrac{\widehat{f}}{\widehat{g}};\ \widehat{f}, \widehat{g} \in F[\sigma],\ \widehat{g}(0) \neq 0 \right\},$$
$$\mathcal{S} = \widehat{\mathcal{D}}_{\widehat{T}} \quad \text{where} \quad \widehat{T} := \left\{ \tfrac{t}{(s-\alpha)^{\deg(t)}};\ t \in T \right\}$$

(compare [3, Def. and Lem. 2.14]). Moreover, the equality

$$\mathcal{D}_T = \mathcal{S}_\sigma := \mathcal{S}_{\{u\sigma^k;\, u \in F\setminus\{0\},\, k \in \mathbb{N}\}}$$

holds, compare [3, Lem. 3.11]. The introduction of α and $\sigma = (s-\alpha)^{-1}$ in this context is due to Pernebo [9].

Note that, if $R \in \mathcal{K}^{k\times\ell}$ is a rational matrix, then its Smith form w.r.t. $\widehat{\mathcal{D}}$ is also the Smith form w.r.t. $\mathcal{S} = \widehat{\mathcal{D}}_{\widehat{T}}$, w.r.t. $\mathcal{D}_T = \mathcal{S}_\sigma$, w.r.t. $F(s)_{\mathrm{pr}}$, and w.r.t. $\mathcal{K}$.

In the sequel we assume that $\mathcal{B}_{1,T}$ is controllable, i.e., $H_1 = P_1^{-1}Q_1$ is the essentially unique left coprime factorization of H_1 over $\mathcal{D}_T$, compare Lemma 0.1. Let

$$H_1 = \widehat{P}_1^{-1}\widehat{Q}_1 = \widehat{N}_1\widehat{D}_1^{-1},$$
$$(\widehat{P}_1,\, -\widehat{Q}_1) \in \widehat{\mathcal{D}}^{p\times(p+m)}, \quad \begin{pmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{pmatrix} \in \widehat{\mathcal{D}}^{(p+m)\times m} \tag{3}$$

denote the left resp. right coprime factorization of H_1 over $\widehat{\mathcal{D}}$. Then there exists a left inverse $(-\widehat{Q}_2^0,\, \widehat{P}_2^0) \in \widehat{\mathcal{D}}^{m\times(p+m)}$ of $\begin{pmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{pmatrix}$.

Theorem 1.3: Let $\mathcal{B}_1$, $\mathcal{B}_2$ be IO behaviors as above, $(\widehat{P}_1,\, -\widehat{Q}_1) \in \widehat{\mathcal{D}}^{p\times(p+m)}$ as in (3), and define $(-\widehat{Q}_2,\, \widehat{P}_2) \in \widehat{\mathcal{D}}^{m\times(p+m)}$ similarly. Then $\mathcal{B}_2$ is a T-stabilizing compensator such that $\mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is proper if and only if

$$\mathcal{S}^{1\times p}(\widehat{P}_1,\, -\widehat{Q}_1) \oplus \mathcal{S}^{1\times m}(-\widehat{Q}_2,\, \widehat{P}_2) = \mathcal{S}^{1\times(p+m)}.$$

Hence, T-stabilizing compensators with proper feedback behavior can be obtained via direct complements of $\mathcal{S}^{1\times p}(\widehat{P}_1,\, -\widehat{Q}_1)$. A parametrization of all such direct complements is given in the next lemma.

Lemma 1.4:

$$\left\{ V \subseteq \mathcal{S}^{1\times(p+m)};\ \mathcal{S}^{1\times p}(\widehat{P}_1,\, -\widehat{Q}_1) \oplus V = \mathcal{S}^{1\times(p+m)} \right\} \cong$$
$$\cong \left\{ (-\widehat{Q}_2,\, \widehat{P}_2) \in \mathcal{S}^{m\times(p+m)};\ (-\widehat{Q}_2,\, \widehat{P}_2)\begin{pmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{pmatrix} = \mathrm{id}_m \right\} =$$
$$= (-\widehat{Q}_2^0,\, \widehat{P}_2^0) + \mathcal{S}^{m\times p}(\widehat{P}_1,\, -\widehat{Q}_1) \cong \mathcal{S}^{m\times p}$$

where $V = \mathcal{S}^{1\times m}(-\widehat{Q}_2,\, \widehat{P}_2)$,
$(-\widehat{Q}_2,\, \widehat{P}_2) = (-\widehat{Q}_2^0,\, \widehat{P}_2^0) + X(\widehat{P}_1,\, -\widehat{Q}_1)$, $X \in \mathcal{S}^{m\times p}$.

The following lemma allows the construction of *proper* compensators $\mathcal{B}_2$:

Lemma 1.5: With the notations from above, let $(-\widehat{Q}_2,\, \widehat{P}_2) := (-\widehat{Q}_2^0,\, \widehat{P}_2^0) + X(\widehat{P}_1,\, -\widehat{Q}_1) \in \mathcal{S}^{m\times(p+m)}$ for some $X \in \mathcal{S}^{m\times p}$. Then the following properties are equivalent:

1) $\widehat{P}_2$ is non-singular and $H_2 = \widehat{P}_2^{-1}\widehat{Q}_2$ is proper.
2) $\det\left(\widehat{P}_2(0)\right) \neq 0$ where $\widehat{P}_2 \in F(\sigma)^{m\times m}$ is considered as rational matrix in $\sigma = \frac{1}{s-\alpha}$, i.e., evaluation at zero signifies evaluation at $\sigma = 0$. Note that $\widehat{P}_2(0) \in F^{m\times m}$ is well-defined since the entries of $\widehat{P}_2$ are contained in $\mathcal{S}$ and hence in $F(s)_{\mathrm{pr}} = F[\sigma]_{F[\sigma]\setminus F[\sigma]\sigma} = \left\{ \frac{\widehat{f}}{\widehat{g}};\ \widehat{f}, \widehat{g} \in F[\sigma],\ \widehat{g}(0) \neq 0 \right\}$ with $\frac{\widehat{f}}{\widehat{g}}(0) = \widehat{f}(0)\widehat{g}(0)^{-1} \in F$.

Theorem 1.6: 1) The IO behavior $\mathcal{B}_1$ admits a T-stabilizing compensator $\mathcal{B}_2$ such that both $\mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$ and $\mathcal{B}_2$ are proper if and only if $\mathcal{B}_{1,T}$ is controllable.

2) Assume that $\mathcal{B}_{1,T}$ is controllable. The polynomial

$$g(T) := \det\left(\widehat{P}_2^0(0) - T\widehat{Q}_1(0)\right) \in F[T]$$

in the indeterminates $T = (T_{ij})_{1\le i\le m, 1\le j\le p}$ is nonzero and hence $g(X_0) = \det(\widehat{P}_2^0(0) - X_0\widehat{Q}_1(0))$ is non-zero for almost all $X_0 \in F^{m\times p}$ since F is of characteristic 0 and thus infinite. Choose one such X_0 and $Y \in \mathcal{S}^{m\times p}$ arbitrarily and define $X := X_0+\sigma Y \in \mathcal{S}^{m\times p}$, $(-\widehat{Q}_2, \widehat{P}_2) := (-\widehat{Q}_2^0, \widehat{P}_2^0) + X(\widehat{P}_1, -\widehat{Q}_1)$, and $H_2 := \widehat{P}_2^{-1}\widehat{Q}_2$. Let $(-Q_{2,\text{cont}}, P_{2,\text{cont}})$ be the controllable realization of H_2 over $\mathcal{D}$, i.e., $H_2 = (P_{2,\text{cont}})^{-1}Q_{2,\text{cont}}$ the left coprime factorization of H_2 over $\mathcal{D}$. Furthermore, choose $A \in \mathcal{D}^{m\times m}$ with $\det(A) \in T$ arbitrarily and define $(-Q_2, P_2) := A(-Q_{2,\text{cont}}, P_{2,\text{cont}})$ and

$$\mathcal{B}_2 := \left\{\left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2\right\}.$$

Then $\mathcal{B}_2$ is proper and a T-stabilizing compensator of $\mathcal{B}_1$ such that $\text{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is also proper. All compensators with these properties are obtained in this fashion. In other words, the triples $(X_0, Y, A) \in F^{m\times p}\times\mathcal{S}^{m\times p}\times\mathcal{D}^{m\times m}$ with $\det(\widehat{P}_2^0(0)-X_0\widehat{Q}_1(0)) \neq 0$ and $\det(A) \in T$ parametrize the set of all T-stabilizing compensators $\mathcal{B}_2$ of $\mathcal{B}_1$ such that both $\mathcal{B}_2$ and $\text{fb}(\mathcal{B}_1, \mathcal{B}_2)$ are proper. Two such triples (X_0, Y, A) and (X_0', Y', A') give rise to the same compensator if and only if $X_0 = X_0'$, $Y_0 = Y_0'$, and A and A' are row equivalent over $\mathcal{D}$.

Lemma 1.7: Assume a behavior $\mathcal{B}_1$ such that $\mathcal{B}_{1,T}$ is controllable as above, with $(\widehat{P}_1, -\widehat{Q}_1)$ and $\left(\begin{smallmatrix} \widehat{N}_1 \\ \widehat{D}_1 \end{smallmatrix}\right)$ defined as in (3). Assume that a T-stabilizing compensator $\mathcal{B}_2$ is constructed as in the previous theorem. Then the transfer matrix

$$H = P^{-1}Q =: \left(\begin{smallmatrix} H_{y_1,u_2} & H_{y_1,u_1} \\ H_{y_2,u_2} & H_{y_2,u_1} \end{smallmatrix}\right)$$

of the feedback behavior $\text{fb}(\mathcal{B}_1, \mathcal{B}_2)$ satisfies

$$H_{y_1,u_2} = \widehat{N}_1\widehat{Q}_2.$$

Corollary 1.8 (Pole placement or spectral assignability): Assume that $\mathcal{B}_1$ is T-stabilizable, i.e., that $\mathcal{B}_{1,T}$ is controllable or that $(P_1, -Q_1)$ admits a right inverse matrix in $\mathcal{D}_T^{(p+m)\times p}$. This signifies that the greatest elementary divisor e_p of $(P_1, -Q_1)$ (w.r.t. $\mathcal{D}$) is contained in T.

1) Define $t_1 := (s-\alpha)e_p$ and let T_1 be the saturated monoid generated by $\langle t_1\rangle$ (in particular $\mathcal{D}_{t_1} = \mathcal{D}_{\langle t_1\rangle} = \mathcal{D}_{T_1}$), i.e., $T_1 = \{t \in \mathcal{D};\ t \neq 0,\ \exists \mu \in \mathbb{N} :\ t|t_1^\mu\}$. Note that T_1 is the smallest saturated monoid containing $(s-\alpha)$ such that $\mathcal{B}_1$ is T_1-stabilizable. By assumption, $T_1 \subseteq T$ and hence $\mathcal{D}_{T_1} \subseteq \mathcal{D}_T$. Theorem 1.6 is also applicable to T_1 instead of T and $\mathcal{S}_1 := \mathcal{D}_{T_1} \cap F(s)_{\text{pr}}$ instead of $\mathcal{S}$, and any triple $(X^0, Y, A) \in F^{m\times p} \times \mathcal{S}_1^{m\times p} \times \mathcal{D}^{m\times m}$ with the properties stated in the previous theorem gives rise to a proper T_1-stabilizing compensator $\mathcal{B}_2$ such that $\text{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is also proper.
2) If, more generally, $A \in \mathcal{D}^{m\times m}$ in 1 is chosen such that $\det(A)$ is in T (but not necessarily in T_1), then $\mathcal{B}_2$ is still a proper T-stabilizing compensator with proper feedback behavior, and $\det\left(\begin{smallmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{smallmatrix}\right) \in \det(A)T_1$.
3) In the situation of 2 assume in particular $F = \mathbb{R}$, a stability region $\Lambda \subseteq \mathbb{C}$ with $\Lambda = \overline{\Lambda}$ and $\alpha \in \Lambda$, and that T is the set of polynomials with zeroes only in Λ. Then

$$\text{ch}(\text{fb}(\mathcal{B}_1, \mathcal{B}_2)^0) \cup \{\alpha\} = \text{V}_{\mathbb{C}}(\det(A)) \cup \text{ch}(\mathcal{B}_1) \cup \{\alpha\}$$

where $\text{ch}(\mathcal{B}_1) := \{\lambda \in \mathbb{C};\ \text{rank}\left((P_1, -Q_1)(\lambda)\right) < p\}$ is the characteristic variety of uncontrollable poles of $\mathcal{B}_1$ etc. and $\text{V}_{\mathbb{C}}(\det(A)) := \{\lambda \in \mathbb{C};\ \det(A)(\lambda) = 0\}$. In other words, the poles of the feedback behavior are the uncontrollable poles of $\mathcal{B}_1$, the elements in $\text{V}_{\mathbb{C}}(\det(A))$ (that can be chosen completely arbitrarily), and possibly another number α.

II. Tracking and Disturbance Rejection

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be IO behaviors as in (1) and (2) such that $\mathcal{B}_2$ is proper and a T-stabilizing compensator of $\mathcal{B}_1$, and $\text{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is proper. Furthermore, we assume a behavior

$$\mathcal{B}_3 = \{w \in \mathcal{F}^p;\ R \circ w = 0\}, \quad R \in \mathcal{D}^{k\times p}.$$

The trajectories of $\mathcal{B}_3$ are the signals that shall be tracked resp. the disturbances that shall be rejected in the following.

Definition 2.1: 1) $\mathcal{B}_2$ is called a *T-tracking compensator* of $\mathcal{B}_1$ for $u_2 \in \mathcal{B}_3$:⇔

$$u_1 = 0,\ u_2 \in \mathcal{B}_3 \quad \Rightarrow \quad e_2 := y_1 + u_2 \ \text{ is } T\text{-small},$$

i.e., if the input u_1 is zero, then the output y_1 T-tracks any tracking signal $(-u_2) \in \mathcal{B}_3$.

2) $\mathcal{B}_2$ is called a *T-(disturbance) rejecting compensator* of $\mathcal{B}_1$ for $u_2 \in \mathcal{B}_3$:⇔

$$u_1 = 0,\ u_2 \in \mathcal{B}_3 \quad \Rightarrow \quad y_1 \ \text{ is } T\text{-small},$$

i.e., any disturbance input $u_2 \in \mathcal{B}_3$ has no significant effect on the output y_1.

Theorem 2.2: Let $\mathcal{B}_2$ be proper and a T-stabilizing compensator of $\mathcal{B}_1$ such that the feedback behavior $\text{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is also proper. With $\mathcal{B}_3$ as above and the notations from Section I, the following assertions hold:

1) $\mathcal{B}_2$ is a T-tracking compensator of $\mathcal{B}_1$ for $u_2 \in \mathcal{B}_3$ if and only if there exists $Z_t \in \mathcal{D}_T^{p\times k}$ such that $\widehat{N}_1\widehat{Q}_2 + \text{id}_p = Z_tR$. In this case $\text{rank}(R) = p$ and $\mathcal{B}_3$ is thus autonomous.
2) $\mathcal{B}_2$ is a T-rejecting compensator of $\mathcal{B}_1$ for $u_2 \in \mathcal{B}_3$ if and only if there exists $Z_d \in \mathcal{D}_T^{p\times k}$ such that $\widehat{N}_1\widehat{Q}_2 = Z_dR$.

Proof: Since the proof of this theorem is very typical for the technique of injective cogenerator quotient signal modules, we explicitly carry out the derivation of the first claim:

By definition, $\mathcal{B}_2$ is a T-tracking compensator of $\mathcal{B}_1$ if and only if

$$\mathcal{B}_{\text{err}} := \{y_1 + u_2 \in \mathcal{F}^p;\ y_1, u_2 \in \mathcal{F}^p,\ R \circ u_2 = 0,\ \exists y_2 \in \mathcal{F}^m :\ P \circ \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) = Q \circ \left(\begin{smallmatrix} u_2 \\ 0 \end{smallmatrix}\right)\}$$

is T-autonomous, i.e.,

$$\mathcal{B}_{\text{err},T} = \{y_1 + u_2 \in \mathcal{F}_T^p;\ y_1, u_2 \in \mathcal{F}_T^p,\ R \circ u_2 = 0,\ \exists y_2 \in \mathcal{F}_T^m :\ P \circ \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) = Q \circ \left(\begin{smallmatrix} u_2 \\ 0 \end{smallmatrix}\right)\} = 0$$

by [2, Thm 1.8] and since $(-)_T$ is an exact functor on behaviors, compare [2, Cor. 1.9]. T-stability of the feedback behavior $\mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$ implies that $P \in \mathrm{Gl}_{p+m}(\mathcal{D}_T)$. Consequently, we can rewrite the previous equality as

$$\{y_1 + u_2 \in \mathcal{F}_T^p;\ y_1, u_2 \in \mathcal{F}_T^p,\ R \circ u_2 = 0,\ \exists y_2 \in \mathcal{F}_T^m : \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) = H \circ \left(\begin{smallmatrix} u_2 \\ 0 \end{smallmatrix}\right)\} = 0,$$

i.e.,

$$\{y_1 + u_2 \in \mathcal{F}_T^p;\ u_2 \in \mathcal{F}_T^p,\ R \circ u_2 = 0,\ y_1 = H_{y_1,u_2} \circ u_2\} = 0.$$

Substituting $H_{y_1,u_2} = \widehat{N}_1 \widehat{Q}_2$ from Lemma 1.7, we obtain

$$\left\{(\widehat{N}_1 \widehat{Q}_2 + \mathrm{id}_p) \circ u_2 \in \mathcal{F}_T^p;\ u_2 \in \mathcal{F}_T^p,\ R \circ u_2 = 0\right\} = 0,$$

i.e.,

$$\{u_2 \in \mathcal{F}_T^p;\ R \circ u_2 = 0\} \subseteq \left\{u_2 \in \mathcal{F}_T^p;\ (\widehat{N}_1 \widehat{Q}_2 + \mathrm{id}_p) \circ u_2 = 0\right\}.$$

Since $\mathcal{F}_T$ is an injective cogenerator over $\mathcal{D}_T$, compare [2, Thm. 1.6], this is equivalent to the existence of $Z_t \in \mathcal{D}_T^{p \times k}$ such that $\widehat{N}_1 \widehat{Q}_2 + \mathrm{id}_p = Z_t R$. ■

By the previous result and Theorem 1.6, there exists a T-tracking resp. T-rejecting compensator $\mathcal{B}_2$ of a given behavior $\mathcal{B}_1$ if and only if the equation

$$M = \widehat{N}_1 X \widehat{P}_1 + Z R \quad \text{where} \qquad (4)$$
$$M := \begin{cases} \widehat{N}_1 \widehat{Q}_2^0 + \mathrm{id}_p & \text{in the case of tracking resp.} \\ \widehat{N}_1 \widehat{Q}_2^0 & \text{in the case of dist. rejection} \end{cases}$$

has a solution $(X, Z) \in \mathcal{S}^{m \times p} \times \mathcal{D}_T^{p \times k}$ such that

$$(-\widehat{Q}_2,\ \widehat{P}_2) := (-\widehat{Q}_2^0,\ \widehat{P}_2^0) + X(\widehat{P}_1,\ -\widehat{Q}_1)$$

has the correct input/output structure and proper transfer matrix. Note that (4) is an inhomogeneous linear equation in the entries of X and Z and can hence be rewritten in the form $(\underline{x},\ \underline{z}) \left(\frac{A}{B}\right) = \underline{m}$ where $\underline{x}$ resp. $\underline{z}$ are $(1 \times (mp))$ resp. $(1 \times (pk))$ matrices containing the entries of X resp. Z etc. Consequently, solvability of (4) can be checked and one solution can be computed (if there are any) by means of Algorithm 5.1 and Algorithm 5.2. Assume a solution $(X^0,\ Z^0) \in \mathcal{S}^{m \times p} \times \mathcal{D}_T^{p \times k}$ of (4) in the following. Let $B^1, \ldots, B^\mu \in \mathcal{S}^{m \times p}$ be a generating system of $\left\{X \in \mathcal{S}^{m \times p};\ \exists Z \in \mathcal{D}_T^{p \times k} : \widehat{N}_1 X \widehat{P}_1 + Z R = 0\right\}$ over $\mathcal{S}$. Such a generating system can also be computed algorithmically. Define the polynomial $g(T) \in F[T]$ in the indeterminates $T = (T_1, \ldots, T_\mu)$ by

$$g(T) := \det\left(\widehat{P}_2^0(0) - \left[X^0(0) + \sum_{i=1}^{\mu} T_i B^i(0)\right] \widehat{Q}_1(0)\right)$$

where all matrices are interpreted as matrices in $F(\sigma) = F(\frac{1}{s-\alpha})$ and evaluation at 0 signifies evaluation at $\sigma = 0$.

Theorem 2.3: For a given plant $\mathcal{B}_1$ and a behavior $\mathcal{B}_3$ and with the notations from above, the following two conditions are equivalent:

1) There exists a T-tracking resp. T-rejecting compensator $\mathcal{B}_2$ of $\mathcal{B}_1$ for signals $u_2 \in \mathcal{B}_3$.
2) a) $\mathcal{B}_1$ is T-stabilizable, i.e., $(P_1,\ -Q_1)$ has a right inverse in $\mathcal{D}_T^{(p+m) \times p}$,
   b) equation (4)
   $$M = \widehat{N}_1 X \widehat{P}_1 + Z R \quad \text{where}$$
   $$M := \begin{cases} \widehat{N}_1 \widehat{Q}_2^0 + \mathrm{id}_p & \text{in the case of tracking} \\ \widehat{N}_1 \widehat{Q}_2^0 & \text{in the case of dist. rej.} \end{cases}$$
   has a solution $(X^0, Z^0) \in \mathcal{S}^{m \times p} \times \mathcal{D}_T^{p \times k}$, and
   c) the polynomial g is non-zero.

Remark 2.4: Condition 2c guarantees the existence of matrices $(-\widehat{Q}_2,\ \widehat{P}_2) = (-\widehat{Q}_2^0,\ \widehat{P}_2^0) + X(\widehat{P}_1,\ -\widehat{Q}_1)$ such that $\widehat{P}_2$ is non-singular and $H_2 := \widehat{P}_2^{-1} \widehat{Q}_2$ is proper, compare Lemma 1.5.

Theorem 2.5: Assume that the conditions of the previous theorem are satisfied. Then all T-tracking resp. T-rejecting compensators are obtained in the following fashion:

1) Let $\tau = (\tau_1, \ldots, \tau_\mu) \in F^\mu$ be a non-zero of the polynomial g, i.e., $g(\tau_1, \ldots, \tau_\mu) \neq 0$. Since $g \neq 0$ and F is an infinite field, almost all $\tau \in F^\mu$ satisfy this condition.
2) Choose arbitrary $\eta_1, \ldots, \eta_\mu \in \mathcal{S}$ and define $\xi_i := \tau_i + \sigma \eta_i$ $(i = 1, \ldots, \mu)$, $X := X^0 + \sum_{i=1}^{\mu} \xi_i B^i$, $(-\widehat{Q}_2,\ \widehat{P}_2) := (-\widehat{Q}_2^0,\ \widehat{P}_2^0) + X(\widehat{P}_1,\ -\widehat{Q}_1)$. Then $\widehat{P}_2$ is non-singular, and $H_2 := \widehat{P}_2^{-1} \widehat{Q}_2 \in F(s)_{\mathrm{pr}}^{m \times p}$.
3) Let $H_2 = (P_{2,\mathrm{cont}})^{-1} Q_{2,\mathrm{cont}}$ be the left coprime factorization of H_2 over $\mathcal{D}$, choose $A \in \mathcal{D}^{m \times m}$ with $\det(A) \in T$ arbitrary, and define $(-Q_2,\ P_2) := A(-Q_{2,\mathrm{cont}},\ P_{2,\mathrm{cont}})$. Then
   $$\mathcal{B}_2 := \left\{\left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2\right\}$$
   is a T-tracking resp. T-rejecting compensator of $\mathcal{B}_1$ for $u_2 \in \mathcal{B}_3$, and all such compensators are obtained in this fashion.

In other terms: The T-tracking resp. T-rejecting compensators are parametrized by

- $\tau \in F^\mu$ such that $g(\tau) \neq 0$,
- $\eta \in \mathcal{S}^\mu$ arbitrary, and
- $A \in \mathcal{D}^{m \times m}$ with $\det(A) \in T$.

Corollary 2.6 (Pole placement): Let T be a saturated multiplicatively closed subset of $\mathcal{D} \setminus \{0\}$ containing $(s - \alpha)$ and assume that $\mathcal{B}_1$ admits a T-tracking resp. T-rejecting compensator $\mathcal{B}_2$, i.e., that the conditions in Theorem 2.3 are satisfied. Define t_1 and T_1 as in Corollary 1.8, i.e., minimally such that $(s - \alpha) \in T_1$, T_1 is saturated, and $\mathcal{B}_1$ is T_1-stabilizable. Algorithm 5.1.2 yields a "minimal" polynomial $t_2 \in \mathcal{D} = F[s]$ such that equation (4) in Theorem 2.3 has a solution $(X,\ Z) \in \mathcal{D}_{t_2}^{m \times p} \times \mathcal{D}_{t_2}^{p \times k}$. By Algorithm 5.2.2 and since $\mathcal{B}_1$ admits T-tracking resp. T-rejecting compensators by assumption, this implies the existence of solutions

$(X, Z) \in \mathcal{S}_3^{m\times p} \times \mathcal{D}_{t_3}^{p\times k}$ where $t_3 := (s-\alpha)\cdot t_2$ and $\mathcal{S}_3 := \mathcal{D}_{t_3} \cap F(s)_{\text{pr}}$.

Let $t' := t_1 t_2$ (or, alternatively, the product of all prime factors appearing in $t_1 t_2$), and denote by T' the saturation of $\langle t' \rangle$.

1) Then $\mathcal{B}_1$ admits a T'-tracking resp. T'-rejecting compensator $\mathcal{B}_2$, and T' is the smallest saturated monoid containing $(s-\alpha)$ with this property.
2) All T'-tracking resp. T'-rejecting compensators $\mathcal{B}_2$ of $\mathcal{B}_1$ can be constructed according to Theorem 2.5 by using T' instead of T, in particular $\mathcal{S}' := \mathcal{D}_{T'} \cap F(s)_{\text{pr}}$ instead of $\mathcal{S}$.
3) Since $\mathcal{B}_1$ admits a T-tracking resp. T-rejecting compensator by assumption, $T' \subseteq T$, $\mathcal{D}_{T'} \subseteq \mathcal{D}_T$, and $\mathcal{S}' \subseteq \mathcal{S}$. If $\mathcal{B}_2 = \left\{ \left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2 \right\}$ is constructed as in 2, but with the parameter $A \in \mathcal{D}^{m\times m}$ having determinant in T and not necessarily in T', then $\mathcal{B}_2$ is a T-tracking resp. T-rejecting compensator of $\mathcal{B}_1$ and $\det \left(\begin{smallmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{smallmatrix}\right) \in \det(A)T'$.
4) In the situation of the previous item, assume in particular the standard situation described in Corollary 1.8.3. Then $\mathcal{B}_2$ is a T-tracking resp. T-rejecting compensator of $\mathcal{B}_1$ and

$$\text{ch}(\text{fb}(\mathcal{B}_1, \mathcal{B}_2)^0) \cup \text{ch}(\mathcal{B}_{\text{err}}) \cup \{\alpha\} = \text{V}_{\mathbb{C}}(\det(A)) \cup \text{V}_{\mathbb{C}}(t')$$

where
$$\mathcal{B}_{\text{err}} := \left\{ y_1 + u_2 \in \mathcal{F}^p;\ y_1 \in \mathcal{F}^p,\ u_2 \in \mathcal{B}_3,\ \exists y_2 \in \mathcal{F}^m : \begin{pmatrix} y_1 \\ y_2 \\ u_2 \\ 0 \end{pmatrix} \in \text{fb}(\mathcal{B}_1, \mathcal{B}_2) \right\}$$

resp.
$$\mathcal{B}_{\text{err}} := \left\{ y_1 \in \mathcal{F}^p;\ \exists u_2 \in \mathcal{B}_3,\ \exists y_2 \in \mathcal{F}^m : \begin{pmatrix} y_1 \\ y_2 \\ u_2 \\ 0 \end{pmatrix} \in \text{fb}(\mathcal{B}_1, \mathcal{B}_2) \right\}$$

in the case of tracking resp. disturbance rejection. Note that $\{\alpha\} \subseteq \text{V}_{\mathbb{C}}(t')$ and $\text{ch}(\mathcal{B}_1) \subseteq \text{V}_{\mathbb{C}}(t_1) \subseteq \text{V}_{\mathbb{C}}(t')$ by construction.

Corollary 2.7: Consider two behaviors

$$\mathcal{B}_t := \{ w \in \mathcal{F}^p;\ R_t \circ w = 0 \}, \quad R_t \in \mathcal{D}^{k_t \times p} \quad \text{and}$$
$$\mathcal{B}_d := \{ w \in \mathcal{F}^p;\ R_d \circ w = 0 \}, \quad R_d \in \mathcal{D}^{k_d \times p}.$$

The problem of constructing T-stabilizing compensators $\mathcal{B}_2$ of a given IO behavior $\mathcal{B}_1$ that *at the same time* T-track signals $u_t \in \mathcal{B}_t$ and T-reject disturbances $u_d \in \mathcal{B}_d$ can also be solved in the present framework: Substitute equation (4) by

$$\begin{pmatrix} \widehat{N}_1 \widehat{Q}_2^0 + \text{id}_p \\ \widehat{N}_1 \widehat{Q}_2^0 \end{pmatrix} = \begin{pmatrix} \widehat{N}_1 \\ \widehat{N}_1 \end{pmatrix} X \widehat{P}_1 + \begin{pmatrix} Z_t & 0 \\ 0 & Z_d \end{pmatrix} \begin{pmatrix} R_t \\ R_d \end{pmatrix}.$$

This equation can be solved algorithmically by means of Algorithm 5.1 and Algorithm 5.2 as well.

All results presented in this section can also be applied for the case of simultaneous T-tracking and T-rejection.

III. Model matching

Consider again the IO behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$ from (1) and (2), as well as the *model behavior*

$$\mathcal{B}_m = \left\{ \left(\begin{smallmatrix} y_m \\ u_2 \end{smallmatrix}\right) \in \mathcal{F}^{p\times p};\ P_m \circ y_m = Q_m \circ u_2 \right\}.$$

Definition 3.1: Assume that $\mathcal{B}_2$ is proper and a T-stabilizing compensator of $\mathcal{B}_1$ such that $\text{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is also proper. We call $\mathcal{B}_2$ a *model matching T-compensator* of $\mathcal{B}_1$ for the model behavior $\mathcal{B}_m$ if $y_1 - y_m$ is T-small whenever

$$\begin{pmatrix} y_1 \\ y_2 \\ u_2 \\ 0 \end{pmatrix} \in \text{fb}(\mathcal{B}_1, \mathcal{B}_2) \quad \text{and} \quad \left(\begin{smallmatrix} y_m \\ u_2 \end{smallmatrix}\right) \in \mathcal{B}_m$$

for some $y_2 \in \mathcal{F}^m$ and $u_2 \in \mathcal{F}^p$.
This signifies that $\mathcal{B}_m$ is T-stable and $H_{y_1,u_2} = H_m$ where H is the transfer matrix of the feedback behavior $\text{fb}(\mathcal{B}_1, \mathcal{B}_2)$ and $H_m = P_m^{-1} Q_m$ is the transfer matrix of $\mathcal{B}_m$. Remember from Lemma 1.7 that $H_{y_1,u_2} = \widehat{N}_1 \widehat{Q}_2$ with the notations of the previous sections.

Theorem 3.2: We consider IO behaviors $\mathcal{B}_1$ and $\mathcal{B}_m$ and the corresponding derived matrices as in the previous sections. Let $r := \text{rank}(\widehat{N}_1)$ and let $U \in \widehat{\mathcal{D}}^{m\times(m-r)}$ be a universal right annihilator of $\widehat{N}_1$ (cf [3, Def. and Lem. 2.7]). Then the following two conditions are equivalent:

1) There exists a model matching T-compensator $\mathcal{B}_2$ of $\mathcal{B}_1$ for the model behavior $\mathcal{B}_m$.
2) a) The matrix $(P_1, -Q_1)$ has a right inverse matrix in $\mathcal{D}_T^{(p+m)\times p}$, i.e., $\mathcal{B}_1$ is T-stabilizable,
   b) The model behavior $\mathcal{B}_m$ is T-stable,
   c) the equation
   $$\widehat{N}_1 \widehat{Q}_2^0 - H_m = \widehat{N}_1 X \widehat{P}_1 \tag{5}$$
   has a solution $X^0 \in \mathcal{S}^{m\times p}$, and
   d) the polynomial $g(T) \in F[T]$ in the indeterminates $T = (T_{ij})_{1\le i\le m-r, 1\le j\le p}$ defined by
   $$g(T) := \det\left(\widehat{P}_2^0(0) - \left[X^0(0) + U(0)T \right] \widehat{Q}_1(0) \right)$$
   is non-zero.

Theorem 3.3: Assume that the conditions of the previous theorem are satisfied. Then all model matching T-compensators are obtained in the following fashion:

1) Let $\tau = (\tau_{ij})_{1\le i\le m-r, 1\le j\le p} \in F^{(m-r)\times p}$ be a non-zero of the polynomial g, i.e., $g(\tau) \neq 0$. Since g is non-zero and F is an infinite field almost all $\tau \in F^{(m-r)\times p}$ satisfy this condition.
2) Choose an arbitrary matrix $\eta \in \mathcal{S}^{(m-r)\times p}$ and define $S := \tau + \sigma\eta \in \mathcal{S}^{(m-r)\times p}$, $X := X^0 + US$, and $(-\widehat{Q}_2, \widehat{P}_2) := (-\widehat{Q}_2^0, \widehat{P}_2^0) + X(\widehat{P}_1, -\widehat{Q}_1)$. Then $\widehat{P}_2$ is non-singular, and $H_2 := \widehat{P}_2^{-1} \widehat{Q}_2 \in F(s)_{\text{pr}}^{m\times p}$.
3) Let $(-Q_{2,\text{cont}}, P_{2,\text{cont}})$ be the controllable realization of H_2 over $\mathcal{D}$, choose an arbitrary matrix $A \in \mathcal{D}^{m\times m}$ with $\det(A) \in T$, and define $(-Q_2, P_2) := A(-Q_{2,\text{cont}}, P_{2,\text{cont}})$. Then
$$\mathcal{B}_2 := \left\{ \left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2 \right\}$$

is a model matching T-compensator of $\mathcal{B}_1$ for the model behavior $\mathcal{B}_m$, and all such compensators can be obtained in this fashion.

In other terms: The model matching T-compensators are parametrized by

- $\tau \in F^{(m-r)\times p}$ such that $g(\tau) \neq 0$,
- $\eta \in \mathcal{S}^{(m-r)\times p}$, and
- $A \in \mathcal{D}^{m\times m}$ with $\det(A) \in T$.

A result on pole placement for model matching T-compensators completely similar to Corollary 2.6 holds as well.

IV. DECOUPLING

We consider again two IO behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$ as in (1) and (2).

Definition 4.1: Assume that $\mathcal{B}_2$ is proper and a T-stabilizing compensator of $\mathcal{B}_1$ such that $\mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is also proper. Then $\mathcal{B}_2$ is called a *decoupling T-compensator* of $\mathcal{B}_1$ if H_{y_1,u_2} is diagonal. Remember that $H_{y_1,u_2} = \widehat{N}_1\widehat{Q}_2$ with the notations of the previous sections.

Obviously, there exists a decoupling T-compensator $\mathcal{B}_2$ of a given behavior $\mathcal{B}_1$ if and only if there is a matrix $X \in \mathcal{S}^{m\times p}$ such that $\widehat{N}_1\widehat{Q}_2^0 - \widehat{N}_1 X\widehat{P}_1$ is a diagonal matrix and $(-\widehat{Q}_2,\, \widehat{P}_2) := (-\widehat{Q}_2^0,\, \widehat{P}_2^0) + X(\widehat{P}_1,\, -\widehat{Q}_1)$ has the correct IO structure and proper transfer matrix. Diagonality of $\widehat{N}_1\widehat{Q}_2^0 - \widehat{N}_1 X\widehat{P}_1$ signifies that $\widehat{N}_1\widehat{Q}_2^0 - \widehat{N}_1 X\widehat{P}_1 = \mathrm{diag}(Z_1, \ldots, Z_p)$ for some $(Z_1, \ldots, Z_p) \in \mathcal{D}_T^p$.

The problem of decoupling can hence be treated completely on the lines of the theory on tracking and disturbance rejection displayed in Section II: Substitute Equation (4) by

$$\widehat{N}_1\widehat{Q}_2^0 = \widehat{N}_1 X\widehat{P}_1 + \mathrm{diag}(Z_1, \ldots, Z_p) \qquad (6)$$

where $X \in \mathcal{S}^{m\times p}$ and $(Z_1, \ldots, Z_p) \in \mathcal{D}_T^{1\times p}$. The existence of solutions $(X,\, (Z_1, \ldots, Z_p)) \in \mathcal{S}^{m\times p} \times \mathcal{D}_T^{1\times p}$ of (6) can again be checked and a parametrization of all such matrices X can be obtained algorithmically, leading to the appropriate definition of a polynomial $g(T) \in F[T]$, $T = (T_1, \ldots, T_\mu)$.

The characterization of the existence of T-tracking resp. T-rejecting compensators in Theorem 2.3 and the constructive parametrization of all such compensators in Theorem 2.5 hold mutatis mutandis for the case of decoupling T-compensators. Also the results on pole placement remain valid.

V. ALGORITHMS

Algorithm 5.1: 1) We cite from [2, Alg. 3.1] (cf. also [12, p. 152, Lem. 4]):
Let R be a principal ideal domain with $\mathrm{quot}(R) =: K$ and let $A \in K^{a\times d}$, $M \in K^{c\times d}$. Let

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = UAV, \quad E = \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_r \end{pmatrix}, \; r = \mathrm{rank}(A),$$

be the Smith form of A with respect to R. Then there exists a matrix $X \in R^{c\times a}$ such that

$$XA = M$$

if and only if

$$\begin{array}{lll} (MV)_{ij}e_j^{-1} \in R & \text{for} & 1 \le i \le c,\; 1 \le j \le r, \\ (MV)_{ij} = 0 & \text{for} & 1 \le i \le c,\; r < j \le d. \end{array}$$

If this is the case, define $\widetilde{X}^1 \in R^{c\times a}$ by

$$\widetilde{X}^1_{ij} := \begin{cases} (MV)_{ij}e_j^{-1} & \text{for} \quad 1 \le j \le r, \\ 0 & \text{for} \quad r < j \le a, \end{cases} \quad 1 \le i \le c.$$

Then $X^1 := \widetilde{X}^1 U \in R^{c\times a}$ satisfies

$$X^1 A = M.$$

Furthermore (parametrization):

$$\left\{X \in R^{c\times a};\; XA = M\right\} \;=\; X^1 + R^{c\times(a-r)}U_2$$

where $U =: \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \in R^{(r+(a-r))\times a}$, i.e., U_2 is a universal left annihilator of A.

2) In particular, consider the ring $R = \mathcal{D}_T = F[s]_T \subseteq K = \mathcal{K} = F(s)$, $T \subseteq \mathcal{D} \setminus \{0\}$ multiplicatively closed, in the previous item. Assume that $XA = M$ has a solution $X \in \mathcal{K}^{c\times a}$, i.e., $(MV)_{ij} = 0$ for $1 \le i \le c$, $r < j \le d$. For $1 \le i \le c$, $1 \le j \le r$, find a representation $(MV)_{ij}e_j^{-1} =: \frac{f_{ij}}{g_{ij}} \in F(s)$, $f_{ij}, g_{ij} \in F[s]$, $\gcd(f_{ij}, g_{ij}) = 1$. Define

$$t_2 := \prod_{i=1}^{c}\prod_{j=1}^{r} g_{ij}$$

and let T_2 be the saturated monoid generated by t_2, i.e., $T_2 = \{t \in \mathcal{D};\, t \neq 0,\, \exists \mu \in \mathbb{N} :\, t | t_2^\mu\}$. Then the equation $XA = M$ has a solution $X \in \mathcal{D}_{T_2}^{c\times a}$, and T_2 is the smallest saturated monoid with this property.

Algorithm 5.2: Consider the ring $\mathcal{D}_T$ for a multiplicatively closed saturated set $T \subseteq \mathcal{D} \setminus \{0\}$. Assume that T contains an element of the form $(s - \alpha)$ for some $\alpha \in F$, define $\sigma := (s - \alpha)^{-1}$ and $\widehat{\mathcal{D}} := F[\sigma]$. Let $A \in F(s)^{a\times d}$, $B \in F(s)^{b\times d}$, $M \in F(s)^{c\times d}$ and assume that the equation

$$XA + ZB = M \qquad (7)$$

has a solution $(X^1, Z^1) \in \mathcal{D}_T^{c\times(a+b)}$. Then, by Algorithm 5.1, the set of all solutions $(X, Z) \in \mathcal{D}_T^{c\times(a+b)}$ of (7) is given by $(X^1, Z^1) + \mathcal{D}_T^{c\times s}(C, D)$ where $s := a+b-\mathrm{rank}\begin{pmatrix} A \\ B \end{pmatrix}$ and $(C, D) \in \widehat{\mathcal{D}}^{s\times(a+b)}$ denotes a universal left annihilator of $\begin{pmatrix} A \\ B \end{pmatrix}$ w.r.t. $\widehat{\mathcal{D}}$ and hence also w.r.t. $\mathcal{D}_T$. Hence,

$$\left\{X \in \mathcal{D}_T^{c\times a};\, \exists Z \in \mathcal{D}_T^{c\times b} :\, (7)\right\} = X^1 + \mathcal{D}_T^{c\times s}C.$$

1) The existence of $Y \in \mathcal{D}_T^{c\times s}$ such that $X^1 + YC$ is proper, i.e., contained in $\mathcal{S}^{c\times a} = (\mathcal{D}_T \cap F(s)_{\mathrm{pr}})^{c\times a}$, can be checked as follows (compare [3, Cor. 3.9 - Cor. 3.14], [2, Alg. 3.2]): Let

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = UCV, \quad E = \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_r \end{pmatrix}, \; r = \mathrm{rank}(C),$$

be the Smith form of C with respect to $\widehat{\mathcal{D}}$. Then there exists $Y \in \mathcal{D}_T^{c\times s}$ such that $X^0 := X^1 + YC$ is proper and thus $(X^0, Z^0) := (X^1, Z^1) + Y(C, D)$ is a solution of (7) in $\mathcal{S}^{c\times a} \times \mathcal{D}_T^{c\times b}$ if and only if

$$(X^1V)_{ij} \text{ is proper} \quad \text{for} \quad 1 \leq i \leq c,\ r < j \leq a.$$

The actual computation of $Y \in \mathcal{D}_T^{c\times s}$ such that $X^1 + YC$ is proper follows from the relation $\mathcal{D}_T = \mathcal{S} + \mathcal{D}_T e_j$, $1 \leq j \leq r$, and can be performed as described in Algorithm 3.13 in [3] or in part 2 of Algorithm 3.2 in [2].

2) Let T_2 be the smallest saturated monoid such that (7) has a solution $(X^1, Z^1) \in \mathcal{D}_{T_2}^{c\times(a+b)}$ according to Algorithm 5.1. Assume furthermore that there exists a matrix Y in $F(s)^{c\times s}$ such that $X^1 + YC$ is proper, i.e., $(X^1V)_{ij}$ is proper for $1 \leq i \leq c$, $r < j \leq a$. Then $T_3 := \left\{(s-\alpha)^k \cdot t \in \mathcal{D};\ k \in \mathbb{N},\ t \in T_2\right\}$ is the smallest saturated monoid containing $(s-\alpha)$ such that (7) has a solution $(X, Z) \in (\mathcal{D}_{T_3} \cap F(s)_{\text{pr}})^{c\times a} \times \mathcal{D}_{T_3}^{c\times b}$.

Remark

The complete derivations and proofs of the results presented here will soon be published in a mathematical journal.

References

[1] P. J. Antsaklis and A. N. Michel. *Linear Systems*. Birkhäuser, Boston, MA, 2006.

[2] I. Blumthaler. Functional T-observers. *Linear Algebra and its applications*, 432:1560–1577, 2010.

[3] I. Blumthaler and U. Oberst. T-observers. *Linear Algebra and its applications*, 430(8–9):2416–2447, 2009.

[4] F. M. Callier and C. A. Desoer. *Multivariable Feedback Systems*. Springer, New York, 1982.

[5] C.-T. Chen. *Linear System Theory and Design*. Harcourt Brace, FortWorth, 1984.

[6] T. Kailath. *Linear systems*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1980.

[7] U. Oberst. Multidimensional constant linear systems. *Acta Appl. Math.*, 20:1–175, 1990.

[8] U. Oberst. Stability and stabilization of multidimensional input/output systems. *SIAM J. Control Optim.*, 45(4):1467–1507, 2006.

[9] L. Pernebo. An algebraic theory for the design of controllers for linear multivariable systems. Feedback realizations and feedback design. *IEEE Trans. Automat. Control*, 26(1):171–194, 1981.

[10] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory. A behavioral approach*. Springer-Verlag, New York, 1998.

[11] A. I. G. Vardulakis. *Linear Multivariable Control. Algebraic analysis and synthesis methods*. John Wiley & Sons Ltd., Chichester, 1991.

[12] M. Vidyasagar. *Control System Synthesis. A factorization approach*. MIT Press, Cambridge, MA, 1985.

[13] W. A. Wolovich. *Linear Multivariable Systems. Applied Mathematical Sciences, Vol. 11*. Springer-Verlag, New York, 1974.