Output Feedback Observers and Control under non-Gaussian Types of Noise

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***Abstract*— The recent applied motivations for mathematical problems on systems and control emphasize increased interest in feedback control under realistic system output information on the basis of observations corrupted by various types of disturbances (noise). The present text deals with several situations of such type where the noise is confined not only to Gaussian descriptions but also allows nonprobabilistic interpretation. The text is restricted to systems with original linear structure which, after being subjected to feedback control, may turn out to become nonlinear. Indicated here are some concise descriptions for an array of problems and results with detailed solution versions to appear. Beginning with description of connections between stochastic Gaussian and set-membership bounding approaches, it further proceeds with problems on output feedback control under control dependent input stochastic noise additively combined with unknown but bounded disturbances. This produces problems under statistical uncertainty. The next item is a discussion of observers under discrete-time measurements that occur at random instants of time subjected to a Poisson distribution. Conditions on system properties and measurement noise are indicated when the produced solution is asymptotically consistent.**

I. The Two Conventional Approaches to Dynamic State Estimation

We are obliged to begin the text with this small earlier unanticipated section on some basic facts that nevertheless may turn out to lie beyond the scope of knowledge of some readers.

It is well-known that the theory of *dynamic state estimation under stochastic noise*, preceded by seminal investigations of N. Wiener and A. N. Kolmogorov [1], [2], emanated from the celebrated Kalman Filter and was broadly developed in the years that followed. On the other hand, many applied motivations produced a demand for problems similar in nature, but treated in a nonprobabilistic setting, under other classes of noise, namely, those that are modeled by *unknown but bounded* functions, with given or unspecified bounds. The estimators (observers) are then presented as *trajectory tubes* that contain the unknown trajectory and based on available measurements. The respective solutions, triggered by papers [3], [4] and book [5], gave rise to the so-called *bounding approach to state estimation* further developed in monographs [6], [7][1].

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[1]Output feedback control under unknown bounds on the uncertain items was developed within the so-called H_∞ theory [8]. Observers for linear systems in the absence of disturbances were introduced in [9].

We therefore proceed further with describing some connections between these two approaches.

II. The Connection Between Observers Under Stochastic and Under Set-Membership Noise.

Given that equations of Kalman filters are known [10], we remind a few facts about the "bounding" solutions. Consider system

$$\dot{x} \in A(t)x + B(t)u + C(t)v, \quad v \in \mathscr{Q}(t), \tag{1}$$

$$y(t) = H(t)x + w, \quad w \in \mathscr{R}(t), \tag{2}$$

$$x(t_0) \in \mathscr{X}_0, \tag{3}$$

where (1) is the system, u is the known control, (2) is the *measurement (observation) equation* and the continuous multivalued functions $\mathscr{Q}(t)$, $\mathscr{R}(t)$, $t \in T = [t_0, \tau]$ with given set $\mathscr{X}_0$ are the given bounds on the unknown input noises $v(t)$, $w(t)$ in the system input, the observations and initial condition $x(t_0)$.

Given the measurement $y = y^*(t)$, $t \in [t_0, \tau]$ the *guaranteed state estimation problem* is to specify at a given time-instant τ the *information set* $X[\tau]$ of all states $x[\tau]$ of system (1) that are consistent with equations (1), (2) when $y(t) \equiv y^*(t)$. In other words, one is to find the cross-sections at time $t = \tau$ of the tube $X[t]$ of trajectories of system (1), emanating from $\mathscr{X}_0$ under some input $v(t)$ under state constraint (2), with $y(t) \equiv y^*(t)$. The problem of finding $X[\tau]$ is further propagated into one of describing the evolution of $X[\tau] = X[\tau, y_\tau(\cdot)]$ in time. The evolution equation for $X[\tau]$ would therefore be a *dynamic set-valued observer* — the "guaranteed filtering" equation for system (1)–(2) under unknown but bounded uncertainties v, w.

Our next question thereby will be to see, whether the equations of the Kalman filter could be also used to describe the information domain $X[\tau, y_\tau(\cdot)]$ for the guaranteed estimation problem of the above. This question is justified by the fact that on the one hand, the tube $X[t] = X[t, y_\tau(\cdot)]$ may be described through the linear-quadratic approximations (see [11]), while on the other — by the well established connections between the Kalman filtering equations and the solutions to the linear-quadratic problem of control.

Take $u = 0$, fix a triplet $\chi(\cdot) = \chi^*(\cdot) = \{v^*(\cdot), w^*(\cdot), x_0^*\}$ with $\chi^*(\cdot) \in \mathscr{Q}(\cdot) \times (y(\cdot) - \mathscr{R}(\cdot)) \times \mathscr{X}_0$ and consider the stochastic differential equations

$$dz = (A(t)z + C(t)v^*(t))dt + \sigma_1(t)d\xi,$$

$$dq = (H(t)z + w^*(t))dt + \sigma_2(t)d\eta,$$

$$x(0) = x_0^* + \zeta, \quad q(0) = 0,$$

where ξ, η are standard normalized Brownian motions with continuous diffusion matrices $\sigma_1(t)$, $\sigma_2(t)$ and $\det(\sigma_i(t)\sigma_i'(t)) \neq 0$ for all $t \in T$, $i = 1, 2$, ζ is a Gaussian vector with zero mean and variance $M^* = \sigma_0\sigma_0'$. Denoting $\sigma_1(t)\sigma_1'(t) = R^*(t)$, $\sigma_2(t)\sigma_2{}'(t) = N^*(t)$ and treating $q = q(t)$ as the available measurement we may find the equations for the minimum variance estimate

$$z^*(t) = \mathrm{E}(z(t) \mid q(s), t_0 \le s \le t)$$

(the respective "Kalman filter").

These are ODE for the estimation

$$dz^* = (A(t)z^* + C(t)v^*(t))dt + \\ + \Sigma(t)H'(t)(N^*(t))^{-1}(dq - (H(t)z^* + w^*(t))dt), \\ z^*(t_0) = x_0^*,$$

and the Riccati equation for the covariance matrix of errors

$$\dot{\Sigma} = A(t)\Sigma + \Sigma A'(t) + R^*(t) - \Sigma H'(t)(N^*(t))^{-1}H(t)\Sigma, \\ \Sigma(t_0) = M^*.$$

It is obvious that the estimate $z^*(t)$ depends on the triplets $\chi^*(\cdot)$ and $\Lambda^* = \Lambda^*(\cdot) = \{M^*, R^*(\cdot), N^*(\cdot)\} \in \Im$.

Consider set

$$Z^*(t) = Z^*(t, \Lambda^*) = \bigcup\{z^*(t) \mid \chi^*(\cdot) \in \mathscr{Q}(\cdot)\times\mathscr{Y}(\cdot)\times\mathscr{X}_0\},$$

where $\mathscr{Y}(t) = y(t) - \mathscr{R}(t)$.

The next assertion is true.

Theorem 2.1: Assume

$$q(t) \equiv \int_{t_0}^{t} y(\tau)d\tau,$$

then

$$X[\tau] = \bigcap\{Z^*(\tau, \Lambda^*) \mid \Lambda^* \in \Im\}.$$

The last result describes a clear connection between the solutions to the linear-quadratic Gaussian filtering problem (the Kalman filter) and the solutions to the deterministic guaranteed state estimation problem for uncertain systems with unknown but bounded "noise" with *hard bounds* on the unknown items.

III. Output Feedback under Control-Dependent Stochastic Noise

We next deal with a class of system models motivated by control problems in automation and financial management. These are subjected to Brownian noise whose parameters depend on the values of the control. The system equation is

$$dx = (A(t)x + B(t)u(t) + C(t)v(t))dt + D(t)dk(u, \xi), \tag{4}$$

$$x(t_0) = x_0 + \zeta,$$

with state vector $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$; $k \in \mathbb{R}^m$, $k_i = u_i d\xi_i$ stands for the control-dependent input noise, $d\xi_i$ — for the pairwise independent normalized Brownian motions with diffusion coefficients σ_i, ζ is a Gaussian vector with zero mean and variance M. The system coefficients are taken continuous.

System (4) is complemented by a measurement equation

$$dy(t) = (H(t)x(t) + w(t))dt + d\eta, \quad y(0) = 0. \tag{5}$$

Here Brownian noise η is with covariance matrix $N(t)$ and the set-membership disturbances v, w, x_0 are similar to the previous section.

Our problem will be to design an output feedback control that would steer the system towards an ellipsoidal target set

$$\mathscr{M} = \mathscr{E}(m, M) = \{x \mid \langle x - m, M(x - m)\rangle \le 1\},$$

where $M = M' > 0$.

The results of the previous section are applicable to (4), (5). Then, under fixed triplet $\chi^*(\cdot) = \{v^*(\cdot), w^*(\cdot), x_0\}$, and given u the Kalman-type filter equations are

$$d\hat{x} = (A(t)x + B(t)u(t) + C(t)v^*(t))dt + \\ + P(t)H(t)'N^{-1}(t)(dy - (H(t)\hat{x} + w^*(t))dt), \\ \hat{x}(t_0) = x^{0*}, \tag{6}$$

$$\dot{P} = A(t)P + PA'(t) + \tfrac{1}{4}D(t)K(u, \sigma)D'(t) - \\ - PH'(t)N^{-1}(t)H(t)P, \\ P(t_0) = P_0. \tag{7}$$

Here

$$K(u, \sigma) = \sum_{i=1}^{m} (\mathbf{e}^{(i)})' \otimes \mathbf{e}^{(i)}\sigma_i^2 u_i^2$$

is a diagonal matrix with elements $\sigma_i^2 u_i^2$ ($\otimes$ stands for the Kronecker product, $\mathbf{e}^{(i)}$ are the unit orths). The *position* (generalized state) of the system is now

$$\{\hat{x}[t] = \hat{x}(t, u), P[t] = P(t, u)\}.$$

An example of a target problem on output feedback control will now be as follows.

Problem 3.1 (OFC): By selecting $u(t, \hat{x}, P)$, steer the ellipsoid $\mathscr{E}(\hat{x}(t), P(t, u))$ from position $\{\hat{x}^0, P(t_0)\}$ towards set $\mathscr{M}$, ensuring $P[\vartheta] = M$, $\|\hat{x}[\vartheta] - m\|^2 \to \min$.

Another version would require to minimize

$$\|P[\vartheta] - M\|^2 + \|\hat{x}[\vartheta] - m\|^2$$

under specified types of matrix norms.

This a problem of controlling *both mean value and covariance matrix* of system (4), (5) through (6) and (7). Mathematically it is the problem of designing the motion of an ellipsoidal-valued tube through feedback control (due to measurement y) of its center $\hat{x}$ and configuration matrix P. The solution also presumes an investigation of related controllability properties for system (6) and (7). Note however that now the theorem on separating estimation and control does not hold. The solution is based on combining methods of papers [12] and [13]. The problem allows two settings — with either unbounded or bounded control.

The mentioned problem may be propagated to one on minmax: minimum over u and maximum over $\{v, w, x_0\}$. Also note that a result similar to Theorem 2.1 is still true for (6), (7).

An important class of problems is related to control under communication constraints, with measurements arriving at fixed or at random instants of time subjected, for example, to Poisson distributions.

IV. Problem with Discrete Observations at Fixed Times

Consider the problem (4) with observation coming at fixed moments τ_i, $i = 1, 2, \ldots, N$, where N may be bounded or tend to ∞:

$$y_i = H_i x(\tau_i) + w_i + \eta_i, \quad w_i \in \mathscr{R}_i,$$

where η_i — Gaussian noise with covariance matrix N_i, and $\mathscr{R}_i$ are balanced convex sets.

We reduce this problem to one considered in the previous section by setting

$$N^{-1}(t) = \sum_{i=1}^{\infty} N_i^{-1}\delta(t - \tau_i),$$

where $\delta(t)$ is the Dirac's delta-function:

$$\delta(t) = 0,\ t \neq 0; \quad \int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0).$$

Then both equations (6)–(7) decompose into continuous and discrete parts. For $t \in (\tau_{i-1}, \tau_i)$ we have

$$d\hat{x} = (A(t)x + B(t)u + C(t)v^*(t))dt, \qquad (8)$$

$$\dot{P} = A(t)P + PA'(t) + \tfrac{1}{4}D(t)K(u,\sigma)D'(t). \qquad (9)$$

At the instant of observation τ_i the center and covariance matrix are updated according to equations[2]

$$\hat{x}(\tau_i + 0) = \qquad (10)$$
$$= \hat{x}(\tau_i - 0) + P(\tau_i + 0)H_i' N_i^{-1}(y_i - (H_i\hat{x}(\tau_i - 0) + w_i^*)),$$

$$P(\tau_i + 0) = P(\tau_i - 0) - \qquad (11)$$
$$- P(\tau_i - 0)H_i'(H_i P(\tau_i - 0)H_i' + N_i)^{-1}H_i P(\tau_i - 0).$$

Now, to construct a measurement feedback control, we have to solve the control problem for filtering equations (8)–(11).

The arrival of a measurement or its absence may also be subjected to some probability. Then the estimate update equations (10), (11) become

$$\hat{x}(\tau_i + 0) = \hat{x}(\tau_i - 0) +$$
$$+ \alpha_i P(\tau_i + 0)H_i' N_i^{-1}(y_i - (H_i\hat{x}(\tau_i - 0) + w_i^*)),$$

$$P(\tau_i + 0) = P(\tau_i - 0) -$$
$$- \alpha_i P(\tau_i - 0)H_i'(H_i P(\tau_i - 0)H_i' + N_i)^{-1}H_i P(\tau_i - 0).$$

where α_i is a Bernoulli random variable with

$$\mathbb{P}\{\alpha_i = 1\} = p_i.$$

In the limit case $\tau_i - \tau_{i-1} = \sigma \to 0$ and $p_i = \lambda\sigma$, this scheme converges to the one with measurements arriving at Poisson times, which is described in the next section.

[2]Note that equations (8) and (9) have an explicit solution.

V. Discrete-Time Observers under Poisson Measurements

Consider system

$$\dot{x} = A(t)x + B(t)u,$$
$$y(t) = H(t)x + w,$$

similar to (1)–(3). Given $y(t)$, $u(t)$, $t \in [t_0, \tau]$, the problem of finding $x(\tau)$ may be transformed to estimating vector x from

$$z(t) = G(t)x + w,$$

with $z(t)$, $G(t)$ given. Here the measurements y are available only at random instants τ_i, where $\{\tau_1, \ldots, \tau_N\}$ is a given Poisson flow, $Pois(\lambda)$ with fixed or unknown but bounded parameter. The formal problem is to indicate time $T = T(N, p, \varepsilon, \delta)$ such that for all $t > T$ the next inequality is true:

$$\mathbb{P}(\operatorname{diam} \mathscr{I}(t) < r + \delta) > 1 - \varepsilon.$$

Here $\mathbb{P}$ stands for probability, $\operatorname{diam} \mathscr{I}(t)$ is the diameter of the convex compact information set $\mathscr{I}(t)$.

Within this section, $w_i = w(\tau_i)$, are the realizations of the bounded measurement noise that is now taken to be uniformly distributed over a balanced bounded convex set $\mathscr{R}$. Such a situation may be relevant for describing signal transmission in computer communication channels.

The problem now is to describe an observer which allows feasible calculation. The solution is given here through the evolution of *set-valued information sets* $\mathscr{I}(t)$, consistent with available information on the unknown parameters of the overall problem given both on-line, through measurements, and also a priori, through the known system coefficients. In contrast with the traditional bounding approach, these information sets now turn out to be random and the related information tube is a piecewise continuous random set-valued function with values in convex compact sets.

In this presentation we investigate the behavior of the mentioned information sets with $t \to \infty$. Namely, we indicate conditions that ensure the diameter of $\mathscr{I}(t)$ to tend to zero (in the absence of noise v) or to a given positive level r (when additional noise v is present). Estimated here is the convergence rate depending on the dimension n of vector x and p of vector y, as well as of the number of measurements received by time t, and a number $\varepsilon > 0$ given in advance. The formal problem is to indicate time $T = T(N, p, \varepsilon, \delta)$ such that for all $t > T$ the next inequality is true:

$$\mathbb{P}(\operatorname{diam} \mathscr{I}(t) < r + \delta) > 1 - \varepsilon.$$

Here $\mathbb{P}$ stands for probability, $\operatorname{diam} \mathscr{I}(t)$ is the diameter of the convex compact information set $\mathscr{I}(t)$.

The related output feedback control has to rely on the described type of dynamic estimators (observers) and presents a computational challenge. The results presented here are a further development of the approach suggested in [14], but deal with a far more general situation.

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