Conditioned-Invariant Polyhedral Sets for Observers with Error Limitation in Discrete-Time Descriptor Systems

José M. Araújo, Péricles R. Barros and Carlos E.T. Dórea

***Abstract*— This works aims to establish a characterization of conditioned-invariant polyhedral sets applied in the context of state estimation in linear discrete-time descriptor systems. It is shown that, assuming causality, the existing conditions for linear systems in the standard form can be extended to descriptor ones, by rewriting the state equation in a suitable form. To this end, a specific descriptor structure for the observer is proposed, whereby limitation of the estimation error can be achieved by the computation of an as small as possible conditioned invariant polyhedron that contains the set of possible initial errors, which is also characterized, together with the corresponding output injection. The effectiveness of methodology is then illustrated by numerical examples.**

I. INTRODUCTION

The state estimation problem plays a central role in control systems theory and applications. The state-feedback technique is many times applicable due the development of state observers. The first important contributions on the theme can be found in the works of Luenberger [1],[2]. In the case of descriptor systems, this issue is a little bit more difficult, due the fact that observability notion differs considerably from the standard systems. Important contributions on observability and observers design can be found in several papers [1],[2],[3],[4],[5],[6],[7]. On the other hand, interest in observers which are able to impose a tight bound on the estimation, coping with disturbances in the plant and measurement noise, is growing in recent times. Some important contribution on disturbance decoupling/attenuattion can be seen in [8]. The use of set invariance [9] for error limitation was introduced in [10],[11] and improved in [12] . In this work, conditioned invariance of polyhedral sets for standard state-space observers is extended to descriptor ones. A descriptor observer is then proposed, with an adequate mixed static plus delayed output injection law that enforces the error to be confined in such an invariant polyhedron. The idea is similar to that used in [13], where the state equation was re-written as a standard one, with constrained control purposes. Numerical examples are presented to illustrate the effectiveness of the proposed observer.

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José M. Araújo is with Universidade Federal de Campina Grande, Departamento de Engenharia Elétrica, Laboratório de Instrumentação e Controle, Rua Aprigio Veloso, S/N, Bodocongó, Campina Grande-PB, on leave of Instituto Federal da Bahia, Departamento de Tecnologia em Eletro-Eletrônica, Rua Emídio dos Santos, S/N, Barbalho, Salvador-BA jomario@ifba.edu.br

Péricles R. Barros is with Universidade Federal de Campina Grande, Departamento de Engenharia Elétrica, Laboratório de Instrumentação eletrônica e Controle, Rua Aprigio Veloso, S/N, Bodocongó, Campina Grande-PB prbarros@ufcg.edu.br

Carlos E.T. Dórea is with Univresidade Federal da Bahia, Departamento de Engenharia Elétrica, Rua Aristides Novis, 2, Federação, Salvador-BA cetdorea@ufba.br

II. Observer with error limitation

A. The observer problem

Consider the state-space model for a perturbed descriptor linear time-invariant system, subject to measurement noise on the output, given by:

$$\begin{aligned} Ex\left(k+1\right) &= Ax\left(k\right) + B_1 d\left(k\right) \\ y(k) &= Cx(k) + \eta(k) \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$ is the state vector, $d \in \mathcal{D} \subset \mathcal{R}^p$ is a bounded disturbance, $y \in \mathcal{R}$ is the output and $\eta \in \mathcal{N} \subset \mathcal{R}$ is a bounded measurement noise. Without loss of generality, consider causal descriptor systems with the following matrices:

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} \tag{2}$$

The problem of obtaining an observer amounts to construct a dynamic system, standard or descriptor, in such a manner that its state vector $\hat{x}(k)$ is such that, for the unperturbed system $(d, \eta \equiv 0)$:

$$\lim_{k\to\infty} \left[x\left(k\right) - \hat{x}\left(k\right)\right] = 0 \tag{3}$$

Several works deal with this problem, and some important contributions can be found in the form of a continuous time standard observer or discrete time descriptor observer [4], descriptor continuous time proportional plus derivative [5], descriptor continuous time proportional plus integral [6] or descriptor discrete time proportional plus integral ones [7].

B. The proposed observer and conditioned invariant sets

Consider the full-order descriptor observer for (1) in the form:

$$\begin{aligned} E\hat{x}\left(k+1\right) &= A\hat{x}\left(k\right) - Ev\left(z(k)\right) - Pv(z(k-1)) \\ \hat{y}\left(k\right) &= C\hat{x}(k) \end{aligned} \tag{4}$$

where $v\left(z\right)$ is a suitable, possibly non-linear, output injection law, with $z = y - \hat{y}$, and

$$P = \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

As already stated, only causal (consequently regular) systems will be considered. A necessary and sufficient condition for this is that matrix (A_{22}) be non-singular [14]. The estimation error dynamics is then given by

$$\begin{aligned} Ee\,(k+1) &= Ae\,(k) + B_1 d(k) + Ev\,(z(k)) + Pv(z(k-1)) \\ z\,(k) &= Ce(k) + \eta(k) \end{aligned} \tag{5}$$

At this point, the following definition for conditioned invariant sets can be stated:

Definition 2.1: Given $0 < \lambda < 1$, the set $\Omega \subset \mathcal{R}^n$, is said to be conditioned-invariant λ-contractive with respect to system (5) if $\forall e(k) \in \Omega$, $\exists v(z(k))$ such that $e(k+1) \in \lambda\Omega, \forall d(k) \in \mathcal{D}, \forall \eta(k) \in \mathcal{N}$.

Now, a standard form for (5) is constructed. The estimation error and the output injection vector can be partitioned as:

$$e\,(k) = \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix},\ v\,(k) = \begin{bmatrix} v_1(z(k)) \\ v_2(z(k)) \end{bmatrix}$$

Henceforth, one can see that:

$$\begin{gathered} \mathrm{e}_1\,(k+1) = A_{11}e_1(k) + A_{12}e_2(k) + \\ \mathrm{B}_{11}d(k) + v_1(z(k)) + A_{12}v_2(z(k-1)), \\ 0 = \mathrm{A}_{21}e_1(k) + A_{22}e_2(k) + B_{12}d(k) + A_{22}v_2(z(k-1)) \end{gathered}$$

By replacing $e_2(k)$ obtained from the algebraic equation, and then advancing it by one step, the following equation comes:

$$e(k+1) = \tilde{A}e(k) + \tilde{B}_1 \begin{bmatrix} d(k) \\ d(k+1) \end{bmatrix} + \varphi(z(k)) \tag{6}$$

where:

$$\tilde{A} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21}) & 0 \end{bmatrix},$$
$$\tilde{B}_1 = \begin{bmatrix} B_{11} - A_{12}A_{22}^{-1}B_{12} & 0 \\ -A_{22}^{-1}A_{21}(B_{11} - A_{12}A_{22}^{-1}) & -A_{22}^{-1}B_{12} \end{bmatrix}$$

We have also that:

$$\varphi(z(k)) = Qv(z(k)), Q = \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & -I \end{bmatrix}$$

Since matrix Q is nonsingular, $v(z(k))$ can be directly obtained from $\varphi(z(k))$.

Let us now consider the following [10],[11],[12]. Disturbance d is supposed to belong to a compact set $\mathcal{D} \subset \mathcal{R}^r$, and noise measurements to the set $\mathcal{N} = \{\eta : |\eta| \leq \overline{\eta}\}$. Ω is a compact set defined on the estimation error space whose interior contains the origin, and it induces the following set of *admissible outputs*:

$$\mathcal{Z}\,(\Omega) = \{z : z = Ce + \eta, e \in \Omega, \eta \in \mathcal{N}\}$$

The set of estimation errors which are consistent with each $z \in \mathcal{Z}\,(\Omega)$ is given by:

$$\mathcal{E}\,(z) = \{e : Ce = z - \eta, \eta \in \mathcal{N}\}$$

In the standard state space form, definition 2.1 can be stated as follows: a set $\Omega \subset \mathcal{R}^n$, is said to be conditioned-invariant λ-contractive with respect to system (6) if $\forall z \in \mathcal{Z}\,(\Omega)\,, \exists\varphi : \tilde{A}e + \tilde{B}_1\bar{d} + \varphi \in \lambda\Omega, \forall\bar{d} = \begin{bmatrix} d & d \end{bmatrix}^T \in \mathcal{V} = \mathcal{D} \times \mathcal{D}, \forall e \in \mathcal{E}(z) \cap \Omega$.

Let now Ω, $\mathcal{D}$ and $\mathcal{N}$ be compact, convex polyhedral sets containing the origin:

$$\begin{gathered} \Omega = \{e : Ge \leq \rho\}\,, \mathcal{D} = \{d : Vd \leq \mu\}, \\ \mathcal{N} = \{\eta\ :\ |\eta| \leq \bar{\eta}\}. \end{gathered}$$

The set of admissible outputs $\mathcal{Z}\,(\Omega)$ is a line segment in $\mathcal{R}$ given by:

$$\mathcal{Z}\,(\Omega) = \{z : z = Ce + \eta, e : Ge \leq \rho, \eta : |\eta| \leq \bar{\eta}\}$$

One can then see that Ω is conditioned-invariant λ-contractive if and only if, $\forall z \in \mathcal{Z}\,(\Omega)$:

$$\begin{gathered} \exists\varphi : G(\tilde{A}e + \tilde{B}_1\bar{d} + \varphi) \leq \lambda\rho, \forall e, \eta : z = Ce + \eta, Ge \leq \\ \rho, |\eta| \leq \bar{\eta}, \forall\bar{d} = \begin{bmatrix} d & d \end{bmatrix}^T : \bar{V}\bar{d} \leq \bar{\mu}, \bar{V} = \\ \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}, \bar{\mu} = \begin{bmatrix} \mu \\ \mu \end{bmatrix} \end{gathered}$$

We now summarize the main results on polyhedral conditioned-invariance presented in [12]. Consider the vectors $\phi(\Omega, z)$ and δ:

$$\begin{gathered} \phi_i(\Omega, z) = \max\nolimits_{e,\eta} G_i\tilde{A}e \\ \text{s.t. } Ge \leq \rho, |\eta| \leq \bar{\eta}, Ce + \eta = z \Leftrightarrow Ge \leq \rho, |Ce - z| \leq \bar{\eta} \end{gathered}$$
$$\begin{gathered} \delta_i = \max\nolimits_{\bar{d}} G_i\tilde{B}_1\bar{d} \\ s.t.\ \bar{V}\bar{d} \leq \bar{\mu} \end{gathered}$$

In terms of these vectors, the conditioned-invariance condition is given by:

$$\exists\varphi(z) : \phi(\Omega, z) + G\varphi(z) \leq \lambda\rho - \delta$$

This condition can be numerically hard to check as function $\phi(\Omega, z)$ is concave, piecewise affine. This difficulty can be overcome by using the external representation for Ω based on its vertices e^j, $j = 1, 2..., n_v$. To each vertex, two break points of $\phi(\Omega, z)$ are associated: $z_-^j = Ce^j - \bar{\eta}$ and $z_+^j = Ce^j + \bar{\eta}$. Then, it is possible to define the discrete set $\mathcal{Z}(\Omega) = \{z : z = z_-^j, z = z_+^j, j = 1, ..., n_v\}$ and its cardinality n_z. The following theorem gives a necessary and sufficient numerically tractable condition for polyhedral conditioned invariance [12]:

Theorem 1: The polyhedron $\Omega = \{Ge \leq \rho\}$ is conditioned-invariant λ-contractive if and only if:

$$\forall l = 1, ..., n_z, \exists\varphi(z^l) : \phi(\Omega, z^l) + G\varphi(z^l) \leq \lambda\rho - \delta$$

This result comes from the fact the $\phi(\Omega, z)$ is affine in the interval $[z^l, z^{l+1}]$. Henceforth, the verification of the condition can be made by solving the following linear-programming problems:

$$\begin{gathered} \epsilon(z^l) = \min\nolimits_{\varepsilon,\varphi} \varepsilon \\ s.t.\ \phi(\Omega, z^l) + G\varphi \leq \varepsilon\rho - \delta \end{gathered} \tag{7}$$

Ω is then conditioned-invariant if and only if $\forall l, \epsilon(z^l) \leq \lambda$.

$$\Lambda = \left\{ x(0^+) \ : \ \exists d(0) : \begin{bmatrix} A_{21} & A_{22} & B_{12} \\ -A_{21} & -A_{22} & -B_{12} \\ T_1G_1 & 0 & 0 \\ 0 & 0 & V \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0^+) \\ d(0) \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ T_1\rho \\ \mu \end{bmatrix} \right\}$$

When the polyhedrons Ω and $\mathcal{D}$ are symmetric with respect to the origin, the following lemma gives a necessary condition for conditioned-invariance of Ω:

Lemma 2.1: $\Omega = e : |Qe| \leq \rho$ is conditioned-invariant λ-contractive only if:

$$\phi(\Omega, 0) \leq \lambda\rho - \delta \tag{8}$$

In this lemma, Although this is only a necessary condition, it is much easier to check than the necessary and sufficient conditions of theorem 2.1, because it does not require vertex computation. Moreover, it turns out to be sufficient as well when $p = n - 1$ (the number of outputs is equal to the number of states minus one) [11].

We are now interested in constructing a conditioned-invariant polyhedron which contains the set of possible initial errors Ω, assumed to be a 0-symmetric convex and compact polyhedron. Ideally, such a polyhedron should be as small as possible in order to impose the tightest limitation on the estimation error. Such a set can be computed by a judicious use of the following algorithms:

Algorithm 1: $X^{k+1} = \text{conv}[\lambda^{-1}R(X^k) \cup X^k]$, with: $R(X^k) = \tilde{A}(\mathcal{E}(0) \cap X^k) + \tilde{B}_1\mathcal{V}$, $X^0 = \Omega$.

Remark 1: In [12] it is shown that the set $X^\infty(\Omega, \lambda) = \lim_{k\to\infty} X^k$ is the minimal symmetric convex set containing Ω which satisfies the necessary condition $\tilde{A}(\mathcal{E}(0) \cap X) + \tilde{B}_1\mathcal{V} \subset \lambda X$ (Lemma 2.1). It is then a candidate to be the minimal conditioned-invariant set that containing Ω. Furthermore, for the particular case where $p = n - 1$ and measurement noise is absent, the set $X^\infty(\Omega, \lambda)$ is actually the minimal conditioned invariant set containing Ω.

In the general case, if $X^\infty(\Omega, \lambda)$ is not invariant, another algorithm (Algorithm 2 in [10] and [12]) can be used, which delivers a small polyhedron, not necessarily the minimal one, which may not even exist in the general case.

With a conditioned-invariant polyhedron at hand, an output injection law that assures error limitation must be computed, and some possibilities are:

1) The online solution, at each step, of the linear programming problem:

$$\begin{gathered} \min_{\varepsilon, \varphi(k)} \varepsilon \\ \text{s.t. } \phi(\Omega, z(k)) + G\varphi(k) \leq \varepsilon\rho - \delta \end{gathered}$$

2) An explicit time-varying, piecewise-affine output injection law computed off-line, in the form:

$$\varphi(z(k), k) = L^j z(k) + \lambda^k w^j$$

where $L^j \in \mathcal{R}^n$ and $w^j \in \mathcal{R}^n$ are constant for $z^j \leq z(k) \leq z^{j+1}$, with $z^j \in \mathcal{Z}(\Omega)$.

Remark 2: For a given conditioned-invariant set Ω, due to the presence of disturbance and measurement noise, only $e(k) \in \Omega$ is assured $\forall k$. However, it is possible to enforce the error to converges to a small conditioned-invariant set $\beta^{-1}\Omega$, $\beta \geq 1$, by a suitable adjust of the output injection law. Further details can be seen in [12].

III. Admissible initial error

A key issue of the proposed observer for error limitation purposes concerns the initialization. The initial state of the systems is unknown, but a reasonable hypothesis is that it belongs to a given region, now characterized. Let us initialize the estimated state as $\hat{x}(0) = 0$, and assume that the initial state of the system belongs to a known region Ψ containing the origin. It can be easily shown by means of a fast-slow decomposition that the state x_1 does not experiment jumps for $k = 0$ [13],[14]. The state partition x_2, however, must be consistent with the algebraic equation:

$$0 = A_{21}x_1(0) + A_{22}x_2(0) + B_{12}d(0)$$

Henceforth, a finite jump can occur if $x_2(0)$ is not consistent with the algebraic equation. The initial state before jump is noted by $x_2(0^-)$ and after jump $x_2(0^+)$. Due to the random nature of the disturbance d a characterization of the set that contains the initial state after jump is necessary. Let the following compact polyhedron characterize the initial state before jump:

$$\Upsilon = \left\{ \begin{bmatrix} x_1(0) \\ x_2(0^-) \end{bmatrix} : \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0^-) \end{bmatrix} \leq \rho \right\} \tag{9}$$

One can then compute a positive projection matrix T_1 using e.g. Fourier-Motzkin elimination technique [15], in order to eliminate G_2, i.e., $T_1G_2 = 0$. The projected polyhedron $T_1G_1x_1 \leq T_1\rho$ can be aggregated to the algebraic equation and to the polyhedron of the disturbance, resulting the set at the top of page. Again, one can eliminate the disturbance d by means of a positive projection matrix T_2, and the final polyhedron obtained is given by:

$$\left\{ \begin{bmatrix} x_1(0) \\ x_2(0^+) \end{bmatrix} : T_2 \begin{bmatrix} A_{21} & A_{22} \\ -A_{21} & -A_{22} \\ T_1G_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0^+) \end{bmatrix} \leq T_2 \begin{bmatrix} 0 \\ 0 \\ T_1\rho \\ \mu \end{bmatrix} \right\}$$

Since the observer is initialized with zeros, the initial error belongs to such polyhedron. Then, for error limitation purposes, a conditioned invariant polyhedron as small as possible that contains Ψ can be computed.

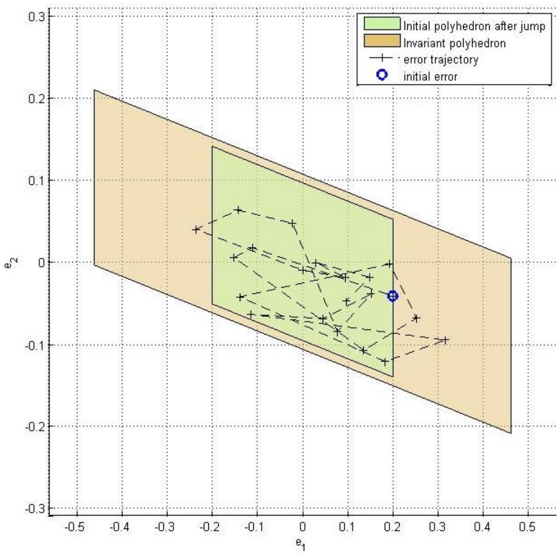

Fig. 1. Polyhedrons for example 1 with an error trajectory

IV. Numerical examples

In this section two examples are given in order to illustrate the concepts discussed in the previous sections. First, consider the descriptor systems with matrices:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -1.1153 & 0.0399 \\ -0.5500 & -2.4828 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -1.1465 \\ 1.1909 \end{bmatrix}, C^T = \begin{bmatrix} 1.1892 \\ -0.0376 \end{bmatrix}$$

The random disturbance is bounded as $|d| \leq 0.2$ and the measurement noise as $|\eta| \leq 0.2$. Considering that before the jump the initial state belongs to the symmetrical polyhedron $|Q|e \leq \rho$, with $Q = I_2$, $\rho = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^T$, the procedure in section III gives the following after jump symmetrical polyhedron:

$$\{G_a e \leq \rho_a\},\ G_a = \begin{bmatrix} 0.2510 & 1.1332 \\ -0.2510 & -1.1332 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}; \rho_a = \begin{bmatrix} 0.1087 \\ 0.1087 \\ 0.2 \\ 0.2 \end{bmatrix}$$

The original polyhedron is not invariant, then the minimal one containing it was computed using the algorithm described in section II, for a contraction rate $\lambda = 0.9$. Also, a linear output injection law is achieved in this case with the gain given by $L = \begin{bmatrix} 0.9387 & -0.2079 \end{bmatrix}^T$.

Figure 1 depicts the polyhedra and an error trajectory initiated in the original polyhedron. In this example, with a linear output injection, one can analyze the closed-loop structure for the error dynamics both in standard and descriptor form. In standard form, the eigenvalues of closed-loop matrix $\tilde{A} + LC$ are both placed in the origin, and the error dynamics is *deadbeat*-like.

Equivalently, if the analysis of the closed-loop behavior is carried out in descriptor form, one can confirm that the closed-loop pencil $zE - (A + EQ^{-1}LC) - z^{-1}PQ^{-1}LC$ has a double root $z = 0$, which is consistent with the analysis in standard form.

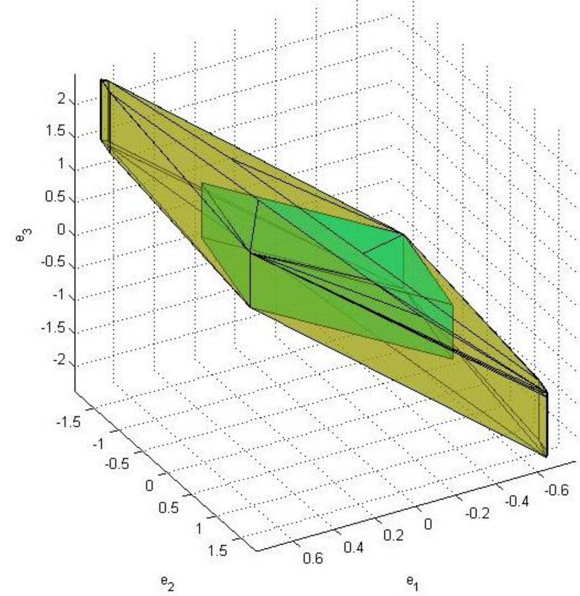

Fig. 2. Polyhedrons for example 2

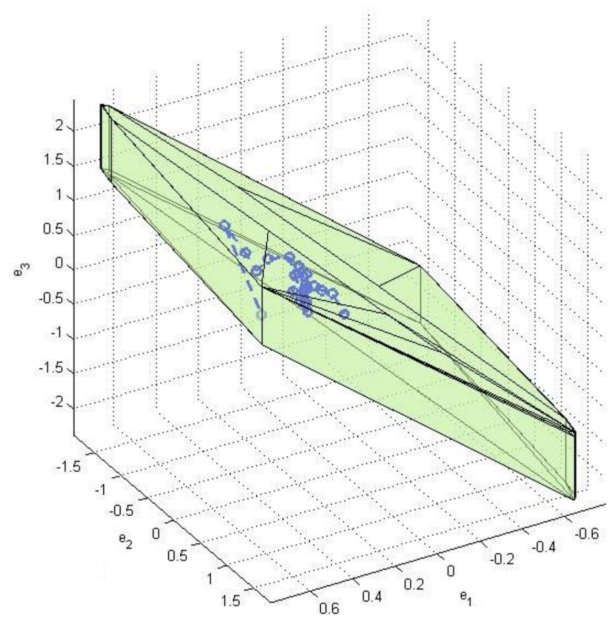

Fig. 3. Invariant polyhedron from example 2 with an error trajectory

As a second example, consider the system given in [13],[16], with $u(k) = 0$. An output is added and the system matrices are:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1.2 & 0 & 0 \\ -1 & -0.7 & -1 \\ 2 & -0.5 & -1.2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, C^T = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Severe disturbance and noise measurement are assumed, with $|d| \leq 0.5$ and $|\eta| \leq 0.5$. Like in the previous example, a symmetrical polyhedron is considered before the jump, with $Q = I_3$, $\rho = \begin{bmatrix} 0.5 & 0.5 & 0.5 \end{bmatrix}^T$. The polyhedron of after jump initial error is given by:

$$\{G_a e \leq \rho_a\},\ G_a = \begin{bmatrix} 1 & -0.25 & -0.6 \\ -1 & 0.25 & 0.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix};\ \rho_a = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

With a contraction rate $\lambda = 0.9$, the invariant polyhedron is shown at Figure 2, together with the original one. A time-variant, piecewise affine output injection law is obtained for this case, given by:

$$v(z(k), k) = \begin{cases} \begin{bmatrix} -0.0640 \\ 0.0376 \\ -0.1240 \end{bmatrix} z(k) + (0.9)^k \begin{bmatrix} -0.0022 \\ -0.0070 \\ 0.0014 \end{bmatrix}, & 0 \leq |z| < 2.6580 \\ \begin{bmatrix} -0.1764 \\ 0.2713 \\ -0.4071 \end{bmatrix} z(k) + (0.9)^k \begin{bmatrix} -0.2073 \\ 0.5363 \\ -0.5565 \end{bmatrix}, & 2.6508 \leq |z| \leq 5.3160 \end{cases}$$

Figure 3 depicts a state trajectory starting from the conditioned-invariant polyhedron, for which the error, as expected, does not escape from within.

V. Conclusion and future works

An extension of set-invariance based techniques for design of observers with error limitation was proposed for discrete-time descriptor systems. Such extension was possible due to a suitable structure for the observer, introduced in this work. A discussion was carried out on the admissibility of initial states, and a characterization of the polyhedron of initial estimation error after jump was given. Future work related to this study concerns $l_\infty - l_\infty$ filtering design and the use of invariant observers for designing output feedback controllers for constrained descriptor systems.

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