Minimum Relative Entropy State Transitions in Linear Stochastic Systems: the Continuous Time Case

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Abstract— **This paper develops a dissipativity theory for dynamical systems governed by linear Itô stochastic differential equations driven by random noise with an uncertain drift. The deviation of the noise from a standard Wiener process in the nominal model is quantified by relative entropy. The paper discusses a dissipation inequality for the noise relative entropy supply. The problem of minimizing the supply required to drive the system between given Gaussian state distributions over a specified time horizon is considered. This problem, known in the literature as the Schrödinger bridge, was treated previously in the context of reciprocal processes. The paper obtains a closed-form smooth solution to a Hamilton-Jacobi equation for the minimum required relative entropy supply by using nonlinear algebraic techniques.**

I. Introduction

We consider a dynamical system whose state is a diffusion process governed by a linear Itô stochastic differential equation (SDE) driven by a random noise. The noise is generated from a standard Wiener process by another SDE with an uncertain drift. The case where the drift vanishes and the noise replicates the Wiener process, represents the *nominal* scenario. A nonzero drift in the noise SDE can be interpreted as the strategy of a hypothetical player who uses the past history of the system state in order to move its probability density function (PDF) away from the *nominal invariant state PDF*. The deviation of the actual noise distribution from the Wiener measure can be quantified by the Kullback-Leibler relative entropy [5]. As a measure of uncertainty in the noise distribution, the relative entropy is often utilized in the robust control of stochastic systems [4], [7], [14], [16].

The *noise relative entropy* over a bounded time interval can be regarded as a stochastic analogue of the supply which is a fundamental concept in the theory of deterministic dissipative systems [18]. This analogy leads to a *dissipation inequality* which links the noise relative entropy supply with the increment in the relative entropy of the state PDF of the system with respect to the nominal invariant state PDF. The *state relative entropy*, therefore, plays the role of a *storage function*. The relative entropy dissipation inequality is related to Jarzynski's equality [11] for the Helmholtz free energy in open dynamical systems. This non-equilibrium thermodynamics viewpoint, where the noise results from interaction of the system with its surroundings (via mechanical work and heat transfer), motivates a stochastic dissipativity theory in the form of a variational problem involving entropy.

Such problems are more complex than their deterministic counterparts since they deal with probability measures (or PDFs) on signal spaces, rather than the signals themselves.

We are mainly concerned with computing the minimum noise relative entropy supply required to drive the system between given initial and terminal state PDFs over a specified time horizon. The *state PDF transition problem*, known as the Schrödinger bridge, was treated previously in a context of reciprocal processes (Markov random fields on the time axis) [1], [3], [6], [12]. This problem was also studied for quantum systems [2], using the formalism of stochastic mechanics [13]. The solution of the Schrödinger bridge problem is related to two coupled integral equations [12, Definition 2.3 on p. 26] and is not available in closed form for a general diffusion model.

We consider the state PDF transition problem with Gaussian initial and terminal state PDFs and undertake a different, somewhat more algebraic, approach. Using Markovization and stochastic linearization of the noise strategy as entropy-decreasing operations, we establish a mean-covariance separation principle which splits the minimum required noise relative entropy supply into two independent terms associated with the mean and covariance matrix of the system state. While the *mean* part is calculated using standard linear quadratic optimization, the *covariance* part (which is a function of matrices) satisfies a Hamilton-Jacobi equation (HJE) complicated by inherent noncommutativity.

This partial differential equation (PDE) has a quadratic Hamiltonian on its right-hand side (with a quadratic nature coming from the diffusion part of the system dynamics) and involves a certain boundary condition. The bilinear "interaction" of solutions of this PDE (which, in the quadratic case, replaces the superposition principle) allows them to be generated in a quasi-additive way. Unlike infinitesimal perturbation techniques based on asymptotic expansions in the small noise limit, our approach provides a *finite* correction scheme which allows a closed-form smooth solution to be found for the HJE.

The correction scheme also employs an ansatz class of "trace-analytic" functions of matrices and a matrix version of the separation of variables which not only copes with the nonlinearity but also essentially "scalarizes" the covariance HJE, thus overcoming the noncommutativity issues. These nonlinear algebraic techniques may therefore be of interest in their own right from the viewpoint of nonlinear PDEs and holomorphic functional calculus.

II. Class of systems being considered

We consider a dynamical system whose state $X := (X_t)_{t \geqslant 0}$ is a diffusion process in $\mathbb{R}^n$ governed by an Itô SDE

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + B\mathrm{d}W_t, \qquad f(x) := \mu + Ax, \quad (1)$$

driven by an $\mathbb{R}^m$-valued random noise $W := (W_t)_{t \geqslant 0}$. Here, $\mu \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, with A Hurwitz and $\mathrm{rank}B = n \leqslant m$, so that the diffusion matrix

$$D := BB^{\mathrm{T}} \qquad (2)$$

is positive definite. The noise W is an Itô process interpreted as an external random noise which originates from interaction of the system with its environment and is generated by another SDE

$$\mathrm{d}W_t = h_t\mathrm{d}t + \mathrm{d}\mathcal{W}_t, \qquad (3)$$

with an uncertain drift $h := (h_t)_{t \geqslant 0}$. Here, h is a random process with values in $\mathbb{R}^m$, adapted to the natural filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0}$ of X, where $\mathcal{F}_t$ is the σ-subalgebra of events induced by the history $X_{[0,t]}$ of X on the time interval $[0, t]$. Also, $\mathcal{W} := (\mathcal{W}_t)_{t \geqslant 0}$ is a m-dimensional standard Wiener process, independent of X_0. Substituting (3) into (1) yields

$$\mathrm{d}X_t = (f(X_t) + Bh_t)\mathrm{d}t + B\mathrm{d}\mathcal{W}_t. \qquad (4)$$

We assume that $\mathbf{E} \int_0^t |h_s|^2 \mathrm{d}s < +\infty$ for all $t > 0$. Together with $D \succ 0$, the local mean square integrability of h ensures the absolute continuity of the system state X_t. The case $h \equiv 0$ (where $W \equiv \mathcal{W}$) represents the *nominal* scenario of the system-environment interaction. A nonzero drift h_t is interpreted as the strategy of a hypothetical player who uses the past history $X_{[0,t]}$ of the system state X_t to move the *state PDF* p_t away from the *nominal invariant state PDF*

$$p_*(x) = (2\pi)^{-n/2}(\det \Pi_*)^{-1/2}\exp(-\|x - \alpha_*\|_{\Pi_*^{-1}}^2/2), \quad (5)$$

which the system would have in the nominal case. The latter is Gaussian, $\mathcal{N}(\alpha_*, \Pi_*)$, with mean α_* and covariance matrix Π_* given by

$$\alpha_* = -A^{-1}\mu, \qquad \Pi_* = \int_0^{+\infty} \mathrm{e}^{At}D\mathrm{e}^{A^{\mathrm{T}}t}\mathrm{d}t, \qquad (6)$$

where Π_* is the infinite-horizon controllability Gramian of the pair (A, B) satisfying the algebraic Lyapunov equation

$$A\Pi_* + \Pi_*A^{\mathrm{T}} + D = 0. \qquad (7)$$

The deviation of the actual noise distribution from the Wiener measure is quantified by

$$E_t := \mathbf{D}(\mathbf{P}_t \| \mathbf{P}_t^*) - \mathbf{D}(\mathbf{P}_0 \| \mathbf{P}_0^*) = \frac{1}{2}\int_0^t \mathbf{E}(|h_s|^2)\mathrm{d}s. \quad (8)$$

Here, Girsanov's theorem [9] is used; $\mathbf{D}(M\|N) := \mathbf{E}_M \ln(\mathrm{d}M/\mathrm{d}N)$ is the Kullback-Leibler relative entropy [5] of a probability measure M with respect to another probability measure N (under the assumption of absolute continuity $M \ll N$); and $\mathbf{P}_t$ and $\mathbf{P}_t^*$ are the restrictions of the true and nominal probability measures $\mathbf{P}$ and $\mathbf{P}_*$ to the σ-algebra $\mathcal{F}_t$. The expectation $\mathbf{E}$ in (8) is over $\mathbf{P}$ under which the noise W, governed by (3), becomes a standard Wiener process if and only if $h \equiv 0$. The *noise relative entropy* E_t over the time interval $[0, t]$ from (8) can be regarded

as a stochastic counterpart of the *supply* in the theory of deterministic dissipative systems [18].

In the nominal case $h \equiv 0$, the state X of the system is a homogeneous Markov diffusion process and the state PDF p_t satisfies the Fokker-Planck-Kolmogorov equation (FPKE)

$$\partial_t p_t = \mathcal{L}^\dagger(p_t), \qquad (9)$$

where

$$\mathcal{L}^\dagger(p) := \mathrm{div}^2(Dp)/2 - \mathrm{div}(fp). \qquad (10)$$

Here, for any twice continuously differentiable function $G := (G_{ij})_{1 \leqslant i,j \leqslant n} : \mathbb{R}^n \to \mathbb{S}_n$, with $\mathbb{S}_n$ the space of real symmetric matrices of order n, we define the maps $\mathrm{div}G : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathrm{div}^2G : \mathbb{R}^n \to \mathbb{R}$ by $\mathrm{div}G := (\sum_{j=1}^n \nabla_j G_{ij})_{1 \leqslant i \leqslant n}$ and $\mathrm{div}^2G := \mathrm{div}\mathrm{div}G = \sum_{1 \leqslant i,j \leqslant n} \nabla_i \nabla_j G_{ij}$, where $\nabla_i := \partial_{x_i}$ is the partial derivative in the ith Cartesian coordinate in $\mathbb{R}^n$. The operator $\mathcal{L}^\dagger$ in (10) is the formal adjoint of the infinitesimal generator $\mathcal{L}$ of X in the nominal case. The action of $\mathcal{L}$ on a twice continuously differentiable test function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with bounded support is described by

$$\mathcal{L}(\varphi) = f^{\mathrm{T}}\nabla\varphi + \mathrm{Tr}(D\varphi'')/2, \qquad (11)$$

where $(\cdot)''$ is the Hessian matrix. Since A is Hurwitz, the system is ergodic under the nominal noise $W = \mathcal{W}$ and the PDF (5) is a steady-state solution of the FPKE (9): $\mathcal{L}^\dagger(p_*) \equiv 0$. The controllability of (A, B) (which follows from $D \succ 0$) is equivalent to $\Pi_* \succ 0$, and is also equivalent to the nonsingularity of the finite-horizon controllability Gramian

$$\Gamma_t := \int_0^t \mathrm{e}^{As}D\mathrm{e}^{A^{\mathrm{T}}s}\mathrm{d}s = \Pi_* - \mathrm{e}^{At}\Pi_*\mathrm{e}^{A^{\mathrm{T}}t} \qquad (12)$$

for any $t > 0$. We define two semigroups of affine transformations $(M_t)_{t \geqslant 0}$ and $(C_t)_{t \geqslant 0}$ by

$$M_t(\alpha) := \mathrm{e}^{At}\alpha + A^{-1}(\mathrm{e}^{At} - I_n)\mu = \alpha_* + \mathrm{e}^{At}(\alpha - \alpha_*), \quad (13)$$

$$C_t(\Sigma) := \mathrm{e}^{At}\Sigma\mathrm{e}^{A^{\mathrm{T}}t} + \Gamma_t = \Pi_* + \mathrm{e}^{At}(\Sigma - \Pi_*)\mathrm{e}^{A^{\mathrm{T}}t}. \quad (14)$$

These semigroups act on $\mathbb{R}^n$ and the set $\mathbb{S}_n^+$ of real positive semi-definite symmetric matrices of order n and describe the nominal evolution of the state mean and covariance matrix

$$\alpha_t := \mathbf{E}X_t, \qquad \Pi_t := \mathbf{cov}(X_t). \qquad (15)$$

The infinitesimal generators of the semigroups are given by

$$\mathcal{M}(\alpha) = \mu + A\alpha, \qquad \mathcal{C}(\Sigma) = A\Sigma + \Sigma A^{\mathrm{T}} + D. \qquad (16)$$

In general (when $h \not\equiv 0$), the linearity of the SDE (4) allows the dynamics of (15) to be described by

$$\dot{\alpha}_t = \mathcal{M}(\alpha_t) + B\beta_t, \quad \dot{\Pi}_t = \mathcal{C}(\Pi_t) + BK_t\Pi_t + \Pi_t K_t^{\mathrm{T}}B^{\mathrm{T}} \quad (17)$$

in terms of the moments

$$\beta_t := \mathbf{E}h_t, \qquad K_t := \mathbf{cov}(h_t, X_t)\Pi_t^{-1}. \qquad (18)$$

III. MARKOVIZATION AND STATE PDF DYNAMICS

For any $t \geqslant 0$, we define a function $\overline{h}_t : \mathbb{R}^n \to \mathbb{R}^m$ associated with the noise strategy h by

$$\overline{h}_t(x) := \mathbf{E}(h_t | X_t = x). \tag{19}$$

In particular, if h_t is a deterministic function of t and the current state X_t, then

$$h_t = \overline{h}_t(X_t), \qquad t \geqslant 0. \tag{20}$$

The noise strategies h, satisfying (20) with probability one, are said to be Markov with respect to the state of the system.

Proposition 1: Suppose that the state PDF $p_t(x)$ of the system, governed by (4) is continuously differentiable in $t > 0$ and twice continuously differentiable in $x \in \mathbb{R}^n$. Then it satisfies the FPKE

$$\partial_t p_t = \operatorname{div}^2(D p_t)/2 - \operatorname{div}((f + B\overline{h}_t)p_t)$$
$$= \mathcal{L}^\dagger(p_t) - \operatorname{div}(B\overline{h}_t p_t), \tag{21}$$

where the operator $\mathcal{L}^\dagger$ is defined by (10).

Proof: In view of the smoothness of p_t and the identity $\mathbf{E}(f(X_t) + Bh_t | X_t = x) = f(x) + B\overline{h}_t(x)$ which follows from (19), the PDE (21) is obtained from the weak formulation of the FPKE for Itô processes in [12, Eqs. (0.11)–(0.13) on p. 21]. $\square$

The PDE (21) governs the PDF of a Markov diffusion process $\xi := (\xi_t)_{t \geqslant 0}$ generated by the SDE $d\xi_t = (f(\xi_t) + B\overline{h}_t(\xi_t))dt + Bd\mathcal{W}_t$. If ξ_0 and X_0 are identically distributed with the state PDF p_0, then ξ_t and X_t share the common PDF p_t for any $t > 0$. The passage $h_t \mapsto \overline{h}_t(X_t)$ from an arbitrary noise strategy h to the Markov one, defined by (19), is referred to as the *Markovization* of h and denoted by $\mathbf{M}$. Although $\mathbf{M}$ preserves the state PDFs p_t of the system, the multi-time probability distributions of X are, in general, modified. They all remain unchanged under the Markovization $\mathbf{M}$ if and only if h is Markov. Since such strategies are invariant under $\mathbf{M}$, the Markovization is idempotent: $\mathbf{M}^2 = \mathbf{M}$.

Theorem 1: The Markovization $\mathbf{M}$ of a noise strategy h does not increase the noise relative entropy supply (8):

$$E_T \geqslant \frac{1}{2} \int_0^T \mathbf{E}(|\overline{h}_t(X_t)|^2)dt, \tag{22}$$

where $\overline{h}_t$ is given by (19). This inequality is an equality if and only if h is Markov in the sense of (20).

Proof: The standard properties of iterated conditional expectations, strict convexity of the squared Euclidean norm $|\cdot|^2$ and Jensen's inequality imply that $\mathbf{E}(|h_t|^2) = \mathbf{E}\mathbf{E}(|h_t|^2|X_t) \geqslant \mathbf{E}(|\overline{h}_t(X_t)|^2)$, where the inequality becomes an equality if and only if (20) holds with probability one. Integration over $t \in [0, T]$ yields (22) whose right-hand side is completely specified by the functions $\overline{h}_t$ and p_t. Since the state PDFs remain unchanged under the Markovization of the noise strategy, then (22) holds as an equality if and only if h is Markov. $\square$

IV. RELATIVE ENTROPY DISSIPATION INEQUALITY

We will now consider the *state relative entropy* defined by

$$R_t := \mathbf{E} \ln q_t(X_t) = \langle p_t, \ln q_t \rangle, \tag{23}$$

where $\langle a, b \rangle := \int_{\mathbb{R}^n} a(x)^\mathrm{T} b(x) dx$ is the inner product of square integrable functions $a, b : \mathbb{R}^n \to \mathbb{R}^r$, and

$$q_t := p_t / p_*. \tag{24}$$

is the true-to-nominal state PDF ratio. Note the difference between R_t and the noise relative entropy E_t defined in (8); see [6, Eq. (3.12) & Remark on p. 321]. In the nominal case, R_t is non-increasing in t, which represents the Second Law of Thermodynamics for homogeneous Markov processes as models of isolated systems [5]. A nonzero h makes (4) an open system and R_t is no longer monotonic.

Theorem 2: For any $T > 0$, the increment $R_T - R_0$ of the state relative entropy (23) satisfies

$$R_T - R_0 \leqslant E_T - \frac{1}{2} \int_0^T \langle p_t, |\overline{h}_t - B^\mathrm{T}\nabla \ln q_t|^2 \rangle dt, \tag{25}$$

where $\overline{h}_t$ and q_t are defined by (19) and (24). This inequality is an equality if and only if h is a Markov noise strategy.

Proof: Differentiation of the right-hand side of (23) gives

$$\begin{aligned}\partial_t R_t &= \langle \partial_t p_t, \ln q_t \rangle + \langle p_t, p_t^{-1}\partial_t p_t \rangle \\ &= \langle \mathcal{L}^\dagger(p_t) - \operatorname{div}(g\overline{h}_t p_t), \ln q_t \rangle. \end{aligned} \tag{26}$$

Here, use is made of (21) and the identities $\partial_t \ln q_t = p_t^{-1}\partial_t p_t$ and $\langle 1, \partial_t p_t \rangle = 0$ which follow from (24) and $\langle 1, p_t \rangle = 1$. Integration by parts reduces (26) to

$$\partial_t R_t = \langle p_t, \mathcal{L}(\ln q_t) \rangle + \langle g\overline{h}_t p_t, \nabla \ln q_t \rangle. \tag{27}$$

In view of Fleming's logarithmic transformation [8] (see, also, [2, Eq. (81) on p. 201]), the operator $\mathcal{L}$, defined by (11), acts on the logarithm of a twice continuously differentiable function $\psi : \mathbb{R}^n \to (0, +\infty)$ as $\mathcal{L}(\ln \psi) = \mathcal{L}(\psi)/\psi - \|\nabla \ln \psi\|_D^2/2$. Application of this relation to $\psi := q_t$ from (24) represents the first inner product in (27) in the form

$$\begin{aligned}\langle p_t, \mathcal{L}(\ln q_t) \rangle &= \langle p_t, \mathcal{L}(q_t)/q_t - \|\nabla \ln q_t\|_D^2/2 \rangle \\ &= \langle p_*, \mathcal{L}(q_t) \rangle - \langle p_t, \|\nabla \ln q_t\|_D^2 \rangle/2 \\ &= -\langle p_t, \|\nabla \ln q_t\|_D^2 \rangle/2, \end{aligned} \tag{28}$$

where $\langle p_*, \mathcal{L}(q_t) \rangle = \langle \mathcal{L}^\dagger(p_*), q_t \rangle = 0$. By substituting (28) into (27), it follows that

$$\begin{aligned}\partial_t R_t &= \langle p_t, \overline{h}_t^\mathrm{T} B^\mathrm{T}\nabla \ln q_t - \|\nabla \ln q_t\|_D^2/2 \rangle \\ &= \langle p_t, |\overline{h}_t|^2 - |\overline{h}_t - B^\mathrm{T}\nabla \ln q_t|^2 \rangle/2, \end{aligned} \tag{29}$$

where the square is completed using (2). Integration of both parts of (29) in t over $[0, T]$ yields

$$R_T - R_0 + \frac{1}{2} \int_0^T \langle p_t, |\overline{h}_t - B^\mathrm{T}\nabla \ln q_t|^2 \rangle dt$$
$$= \frac{1}{2} \int_0^T \mathbf{E}(|\overline{h}_t(X_t)|^2)dt \leqslant E_T, \tag{30}$$

where the inequality replicates (22). Now, (25) follows from (30). The claim that (25) holds as an equality if and only if h is Markov, follows from the second part of Theorem 1. $\square$

The relation (25) can be regarded as a *dissipation inequality* [18, pp. 327, 348], with the state relative entropy (23)

playing the role of a *storage function*. Thus, the absolute continuity of the state distribution of the system can only be destroyed within the finite time T using an infinite noise relative entropy supply E_T. Note also that (25), which becomes an equality for Markov noise strategies, can be thought of as an analogue of Jarzynski's equality from nonequilibrium thermodynamics [11].

V. GAUSSIAN STATE PDF TRANSITION

We will now consider the problem of driving the system (4) from a Gaussian initial state PDF $p_0 := \sigma$ to a Gaussian terminal state PDF $p_T := \theta$ at a specified time $T > 0$ so as to minimize the supply (8) over the interval $[0, T]$:

$$J_T(\sigma, \theta) := \inf\{E_T : p_0 = \sigma, \ p_T = \theta\}. \quad (31)$$

Here,

$$\sigma \sim \mathcal{N}(\alpha_0, \Pi_0), \qquad \theta \sim \mathcal{N}(\alpha_T, \Pi_T), \quad (32)$$

with $\Pi_0, \Pi_T \succ 0$. The required supply (31) vanishes if and only if θ is *nominally reachable* from σ in time T in the sense that $\theta = e^{\mathcal{L}^\dagger T}(\sigma)$. Here, $e^{\mathcal{L}^\dagger T}$ is the linear integral operator (with a Markov transition kernel) which relates the terminal state PDF p_T of the system at time T with the initial state PDF p_0 under the nominal FPKE (9). In particular, $J_T(p_*, p_*) = 0$ and, more generally, $J_T(\mathcal{N}(\alpha, \Sigma), \mathcal{N}(M_T(\alpha), C_T(\Sigma))) = 0$. In (31), the intermediate state PDFs p_t, with $0 < t < T$, are not required to be Gaussian. Nevertheless, we can restrict attention to noise strategies which are not only Markov, but are also affine with respect to the state of the system, thus making the intermediate state PDFs also Gaussian; see Fig. 1.
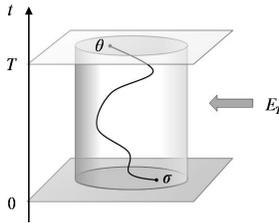

Fig. 1. The cylinder represents the set $\mathbb{G} \times [0, T]$ whose base $\mathbb{G}$ is formed by nonsingular Gaussian PDFs in $\mathbb{R}^n$. The minimization of the noise relative entropy supply E_T in driving the system state PDF between $\sigma, \theta \in \mathbb{G}$ in time T can be restricted to affine Markov noise strategies without affecting the minimum. Such strategies generate paths of Gaussian state PDFs from σ to θ which are entirely contained by the cylinder.

VI. MEAN-COVARIANCE SEPARATION PRINCIPLE

For any $T > 0$, we define a function $S_T : \mathbb{P}_n^2 \to \mathbb{R}_+$, with $\mathbb{P}_n$ the set of real positive definite symmetric matrices of order n, by

$$S_T(\Sigma, \Theta) := \inf \int_0^T \text{Tr}(K_t \Pi_t K_t^{\text{T}}) dt. \quad (33)$$

Here, the minimization is over $\mathbb{R}^{m \times n}$-valued functions K_t from (18) such that the state covariance matrix Π_t, governed by the second of the ordinary differential equations (ODEs) (17) and initialized at $\Pi_0 := \Sigma$, satisfies the terminal condition $\Pi_T = \Theta$. Since $C_T(\Sigma)$ is reachable from Σ in

time T with $K_t \equiv 0$ by the action of the nominal state covariance semigroup (14), then

$$S_T(\Sigma, C_T(\Sigma)) = 0. \quad (34)$$

Theorem 3: The minimum required noise relative entropy supply in (31)–(32) can be computed as

$$J_T(\sigma, \theta) = (\|\alpha_T - M_T(\alpha_0)\|_{\Gamma_T^{-1}}^2 + S_T(\Pi_0, \Pi_T))/2, \quad (35)$$

where (12), (13), (33) are used. The optimal noise strategy is an affine function of the current state of the system:

$$h_t = \beta_t + K_t(X_t - \alpha_t), \quad (36)$$

where K_t is a function delivering the minimum in (33), and

$$\beta_t := B^{\text{T}} e^{A^{\text{T}}(T-t)} \Gamma_T^{-1}(\alpha_T - M_T(\alpha_0)), \quad 0 \leqslant t \leqslant T, \quad (37)$$

Proof: It follows from (17) that the state mean α_t is completely specified by the initial condition α_0 and the function β_t from (18). Similarly, the state covariance matrix Π_t is completely specified by Π_0 and the function K_t. The relative entropy supply (8) affords the lower bound:

$$2E_T = \int_0^T \mathbf{E}(|h_t|^2) dt \geqslant \int_0^T (|\beta_t|^2 + \text{Tr}(K_t \Pi_t K_t^{\text{T}})) dt. \quad (38)$$

Indeed, by [10, Theorem 7.7.7 on p. 473], the Schur complement of the block $\mathbf{cov}(X_t)$ in the joint covariance matrix of X_t and h_t is positive semi-definite. Hence, $\mathbf{cov}(h_t) \succcurlyeq \mathbf{cov}(h_t, X_t)(\mathbf{cov}(X_t))^{-1}\mathbf{cov}(X_t, h_t) = K_t \Pi_t K_t^{\text{T}}$ in view of (15) and (18), and $\mathbf{E}(|h_t|^2) = |\beta_t|^2 + \text{Tr}\mathbf{cov}(h_t) \geqslant |\beta_t|^2 + \text{Tr}(K_t \Pi_t K_t^{\text{T}})$, which implies (38). This inequality becomes an equality if and only if h_t is related to X_t by the affine map (36) with probability one. In view of (15) and (18), the passage from the original noise strategy h to the right-hand side of (36) describes a *stochastic linearization* of h. We denote this operation by $\mathbf{L}$. By construction, it is idempotent: $\mathbf{L}^2 = \mathbf{L}$. The stochastic linearization $\mathbf{L}$ yields an affine noise strategy which is not only Markov in the sense of (20), but also preserves the first two moments of X. Under the affine noise strategy, the state process X is Gaussian, provided that the initial state X_0 is Gaussian. Thus, if h is a noise strategy which drives the state PDF of the system from σ to θ, described by (32), in time T, then $\widehat{h} := \mathbf{L}(h)$ is an affine Markov strategy under which X has the same mean and covariance matrix as it does under h. By the latter property, $\widehat{h}$ also drives the state PDF of the system to the Gaussian PDF θ, but supplying the same or smaller noise relative entropy to the system over $[0, T]$ in view of (38). Therefore, the minimization of E_T can be reduced, without changing the minimum, to a minimization over affine Markov noise strategies (36). Hence, recalling (32) and (33),

$$2J_T(\sigma, \theta) = \inf \int_0^T (|\beta_t|^2 + \text{Tr}(K_t \Pi_t K_t^{\text{T}})) dt$$

$$= S_T(\Pi_0, \Pi_T) + \inf \int_0^T |\beta_t|^2 dt. \quad (39)$$

Here, the first infimum is over the functions β_t and K_t from (18) such that the state mean α_t and covariance matrix Π_t, governed by the ODEs (17) with initial conditions α_0 and Π_0, satisfy the terminal conditions specified by α_T and Π_T, whereas the second infimum is only concerned with β_t. By the first of the ODEs (17), the boundary conditions

on the state mean are equivalent to $\int_0^T e^{A(T-t)}B\beta_t dt = \alpha_T - M_T(\alpha_0)$, where (13) is used. The second infimum in (39) is found by solving the linearly constrained quadratic optimization problem and achieved at the function β_t from (37), with $\min \int_0^T |\beta_t|^2 dt = \|\alpha_T - M_T(\alpha_0)\|_{\Gamma_T^{-1}}^2$, which yields (35). $\square$

The representation (35) splits J_T into two independently computed terms which are associated with the mean and the covariance matrix of the system state. This decoupling is similar to the separation principle of LQG control.

VII. COVARIANCE HAMILTON-JACOBI EQUATION

We will now consider the "covariance" part S_T of the minimum required supply J_T from (35) whose "mean" part is already computed in Theorem 3.

Lemma 1: Suppose that $S_T(\Sigma, \Theta)$ from (33) is smooth with respect to $T > 0$ and $\Sigma \succ 0$. Then it satisfies the HJE

$$\partial_T S_T = F(\Sigma, \partial_\Sigma S_T), \quad F(\Sigma, \Phi) := \mathrm{Tr}((\mathcal{C}(\Sigma) - D\Phi\Sigma)\Phi), \quad (40)$$

where (2) and (16) are used.

Proof: From (17) and (33), it follows that, under the assumption of smoothness, S_T satisfies a Hamilton-Jacobi-Bellman equation

$$\partial_T S_T = \mathrm{Tr}(\mathcal{C}(\Sigma)\Phi_T)$$
$$+ \min_{K \in \mathbb{R}^{m \times n}} (\mathrm{Tr}(K\Sigma K^{\mathrm{T}}) + \mathrm{Tr}((BK\Sigma + \Sigma K^{\mathrm{T}}B^{\mathrm{T}})\Phi_T)), \quad (41)$$

where $\Phi_T := \partial_\Sigma S_T$. Since $K\Sigma K^{\mathrm{T}} + K\Sigma\Phi B + B^{\mathrm{T}}\Phi\Sigma K^{\mathrm{T}} = (K + B^{\mathrm{T}}\Phi)\Sigma(K^{\mathrm{T}} + \Phi B) - B^{\mathrm{T}}\Phi\Sigma\Phi B$ and $\Sigma \succ 0$, the minimum in (41) is only achieved at $K = -B^{\mathrm{T}}\Phi_T$ and is equal to $-\mathrm{Tr}(D\Phi_T\Sigma\Phi_T)$, which implies (40). $\square$

VIII. CORRECTION SCHEME

The Hamiltonian F in (40) is quadratic in its second argument. Hence, if $\widehat{S}_T$ satisfies the covariance HJE, then

$$S_T := \widehat{S}_T + \widetilde{S}_T \quad (42)$$

is also a solution of (40) if and only if $\widetilde{S}_T$, playing the role of a correcting function, satisfies a modified HJE

$$\partial_T \widetilde{S}_T = \widehat{F}(\Sigma, \partial_\Sigma \widetilde{S}_T). \quad (43)$$

Here, $\widehat{F}$ is obtained by correcting the Hamiltonian F by a term arising from the bilinear "interaction" between $\widehat{S}_T$ and $\widetilde{S}_T$:

$$\widehat{F}(\Sigma, \Phi) := F(\Sigma, \Phi) - 2\mathrm{Tr}(D(\partial_\Sigma \widehat{S}_T)\Sigma\Phi)$$
$$= \mathrm{Tr}((\widehat{\mathcal{C}}(\Sigma) - D\Phi\Sigma)\Phi), \quad (44)$$

with

$$\widehat{\mathcal{C}}(\Sigma) := (A - D\partial_\Sigma \widehat{S}_T)\Sigma + \Sigma(A - D\partial_\Sigma \widehat{S}_T)^{\mathrm{T}} + D \quad (45)$$

obtained by modifying the matrix A in the infinitesimal generator $\mathcal{C}$ from (16). The operator $\widehat{\mathcal{C}}$ depends parametrically on T and Σ through $\widehat{S}_T$.

IX. STARTING SOLUTION

Ignoring, for the moment, the boundary condition (34), we will find a particular solution $\widehat{S}_T$ of the covariance HJE (40) as a starting point for the correction scheme (42)–(45). Note that application of the Hamiltonian F from (40) to the function $\Phi := \partial_\Sigma \ln \det \Sigma = \Sigma^{-1}$ yields a constant: $F(\Sigma, \Sigma^{-1}) = \mathrm{Tr}((A\Sigma + \Sigma A^{\mathrm{T}} + D - D\Sigma^{-1}\Sigma)\Sigma^{-1}) = 2\mathrm{Tr}A$. Furthermore, the class of those smooth functions $S_T(\Sigma, \Theta)$, which are affine with respect to Σ, is closed under ∂_T and the action of the Hamiltonian $S_T \mapsto F(\Sigma, \partial_\Sigma S_T)$. Finally, solutions of the quadratic HJE (40) do not obey the superposition principle, and the bilinear interaction of $\ln \det \Sigma$ and an affine function S_T is described by $\mathrm{Tr}(D\Sigma^{-1}\Sigma\partial_\Sigma S_T) = \mathrm{Tr}(D\partial_\Sigma S_T)$ which is also constant in Σ. These observations suggest looking for a particular solution of (40) in the class of functions

$$\widehat{S}_T(\Sigma, \Theta) = \ln \det \Sigma + \mathrm{Tr}(\Sigma\Xi_T(\Theta)) + \Upsilon_T(\Theta). \quad (46)$$

Here, $\Xi_T(\Theta)$ and $\Upsilon_T(\Theta)$ are smooth functions of the time horizon $T > 0$ and the terminal state covariance matrix Θ with values in $\mathbb{S}_n$ and $\mathbb{R}$, respectively, with $\det \Xi_T(\Theta) \neq 0$.

Lemma 2: The following function is a particular solution of the covariance HJE (40) in the class (46):

$$\widehat{S}_T(\Sigma) := \ln \det U_T(\Sigma) + \mathrm{Tr}U_T(\Sigma). \quad (47)$$

Here, the map $U_T : \mathbb{P}_n \to \mathbb{P}_n$ is defined using (12), (14) as

$$U_T(\Sigma) := \Gamma_T^{-1/2}C_T(\Sigma)\Gamma_T^{-1/2} - I_n. \quad (48)$$

Proof: By differentiating the ansatz (46), it follows that

$$\partial_T \widehat{S}_T = \mathrm{Tr}(\Sigma\partial_T\Xi_T) + \partial_T\Upsilon_T, \quad \partial_\Sigma \widehat{S}_T = \Sigma^{-1} + \Xi_T. \quad (49)$$

In view of (16), substitution of $\partial_\Sigma \widehat{S}_T$ into (40) yields

$$F(\Sigma, \partial_T \widehat{S}_T)$$
$$= \mathrm{Tr}((A\Sigma + \Sigma A^{\mathrm{T}} + D - D(\Sigma^{-1} + \Xi_T)\Sigma)\partial_\Sigma \widehat{S}_T)$$
$$= \mathrm{Tr}((A\Sigma + \Sigma A^{\mathrm{T}} - D\Xi_T\Sigma)(\Sigma^{-1} + \Xi_T))$$
$$= \mathrm{Tr}((\Xi_T A + A^{\mathrm{T}}\Xi_T - \Xi_T D\Xi_T)\Sigma) + \mathrm{Tr}(2A - D\Xi_T). \quad (50)$$

By equating $\partial_T \widehat{S}_T$ from (49) with the right-hand side of (50), it follows that the ansatz function $\widehat{S}_T$ in (46) is a solution of (40) if and only if Ξ_T and Υ_T satisfy the ODEs

$$\partial_T \Xi_T = \Xi_T A + A^{\mathrm{T}}\Xi_T - \Xi_T D\Xi_T, \quad (51)$$
$$\partial_T \Upsilon_T = \mathrm{Tr}(2A - D\Xi_T). \quad (52)$$

Multiplication of the right-hand side of (51) by Ξ_T^{-1} yields a matrix whose trace coincides with the right-hand side of (52). Hence, $\partial_T \ln |\det \Xi_T| = \mathrm{Tr}(\Xi_T^{-1}\partial_T\Xi_T) = \partial_T\Upsilon_T$, and

$$\Upsilon_T = \ln |\det \Xi_T| + \Upsilon_0 - \ln |\det \Xi_0|. \quad (53)$$

Left and right multiplication of both parts of (51) by Ξ_T^{-1} yields a differential Lyapunov equation

$$\partial_T(\Xi_T^{-1}) = -\Xi_T^{-1}(\partial_T\Xi_T)\Xi_T^{-1} = D - A\Xi_T^{-1} - \Xi_T^{-1}A^{\mathrm{T}} \quad (54)$$

for Ξ_T^{-1} with a unique equilibrium point $\Xi_T = -\Pi_*^{-1}$; cf. (7). The general solution of (54) is expressed via (12) as

$$\Xi_T = e^{A^{\mathrm{T}}T}(\Xi_0^{-1} + \Gamma_T)^{-1}e^{AT}. \quad (55)$$

Setting $\Upsilon_0 := \ln |\det \Xi_0|$ in (53) and $\Xi_0 \to \infty$ in (55) gives

$$\Xi_T = e^{A^{\mathrm{T}}T}\Gamma_T^{-1}e^{AT}, \quad \Upsilon_T = 2T\mathrm{Tr}A - \ln \det \Gamma_T. \quad (56)$$

Substitution of (56) into (46) yields the particular solution of (40) described by (47)–(48). $\square$

X. TRACE-ANALYTIC CORRECTION

Since $\widehat{S}_T \to -\infty$ as $\Sigma \to 0$, regardless of the choice of Ξ_T and Υ_T, the class (46) can not provide a nonnegative solution to the HJE (40). We will therefore correct $\widehat{S}_T$ from (47) by adding a function $\widetilde{S}_T := \widetilde{S}_T(\Sigma, \Theta)$ such that (42) is a nonnegative solution of (40) satisfying the boundary condition (34). Substitution of $\partial_\Sigma \widehat{S}_T$ from (49) into the correction scheme (43)–(45) yields

$$\partial_T \widetilde{S}_T = \text{Tr}((\overbrace{(A - D\Xi_T)\Sigma + \Sigma(A - D\Xi_T)^{\text{T}} - D}^{\widehat{\mathcal{C}}(\Sigma)} \\ - D(\partial_\Sigma \widetilde{S}_T)\Sigma)\partial_\Sigma \widetilde{S}_T). \quad (57)$$

To find the correcting function $\widetilde{S}_T$, as a solution to (57) such that S_T in (42) is nonnegative and satisfies (34), we will employ yet another ansatz:

$$\widetilde{S}_T := \text{Tr}\chi(\Omega_T) + \rho_T, \qquad \Omega_T := \Psi_T \Sigma \Psi_T. \quad (58)$$

Here, χ is a nonconstant function of a complex variable, analytic in a neighbourhood of $\mathbb{R}_+$ and real-valued on $\mathbb{R}_+$. Also, $\Psi_T(\Theta)$ and $\rho_T(\Theta)$ are smooth functions of the time horizon T and the terminal state covariance matrix Θ with values in $\mathbb{P}_n$ and $\mathbb{R}$, respectively. The matrix Ψ_T, which specifies the linear operator $\Sigma \mapsto \Omega_T$ in (58), enters the "trace-analytic" function $\text{Tr}\chi(\Omega_T)$ only through Ψ_T^2, since

$$\text{Tr}\chi(\Omega_T) = \text{Tr}\chi(\Psi_T^{-1}\Psi_T^2 \Sigma \Psi_T) = \text{Tr}\chi(\Psi_T^2 \Sigma), \quad (59)$$

where we use the invariance of the trace under similarity transformations and their commutativity with analytic functions of matrices. To find suitable χ, Ψ_T, ρ_T, we compute the derivatives $\dot{\widetilde{S}} = \text{Tr}((\dot{\Psi}\Psi^{-1} + \Psi^{-1}\dot{\Psi})\chi'(\Omega)\Omega) + \dot{\rho}$ and $\partial_\Sigma \widetilde{S} = \Psi\chi'(\Omega)\Psi$ by using Lemma 4 and its corollary (A2)–(A3) in Appendix A, with the subscript T omitted for brevity, and $(\dot{}) := \partial_T$. Substituting the derivatives into (57) yields

$$\text{Tr}((\dot{\Psi}\Psi^{-1} + \Psi^{-1}\dot{\Psi})\chi'(\Omega)\Omega) + \dot{\rho}$$
$$= \text{Tr}((\widehat{\mathcal{C}}(\Sigma) - D\Psi\chi'(\Omega)\Psi\Sigma)\Psi\chi'(\Omega)\Psi)$$
$$= \text{Tr}((\Psi(A - D\Xi)\Psi^{-1} + \Psi^{-1}(A - D\Xi)^{\text{T}}\Psi)\chi'(\Omega)\Omega) \\ - \text{Tr}(\Psi D\Psi\chi'(\Omega)(I_n + \chi'(\Omega)\Omega)), \quad (60)$$

where use is made of $\Omega = \Psi\Sigma\Psi$ from (58) and the commutativity of $\chi'(\Omega)$ and Ω. Regrouping the terms of (60) yields

$$\sum_{k=1}^{2} \text{Tr}(G_k \chi_k(\Omega_T)) + \partial_T \rho_T = 0 \quad (61)$$

for all $T > 0$ and $\Sigma, \Theta \succ 0$. Here,

$$G_1(T, \Theta) := \Psi_T D \Psi_T, \quad (62)$$
$$G_2(T, \Theta) := (\partial_T \Psi_T)\Psi_T^{-1} + \Psi_T^{-1}\partial_T \Psi_T$$
$$- \Psi_T(A - D\Xi_T)\Psi_T^{-1} - \Psi_T^{-1}(A - D\Xi_T)^{\text{T}}\Psi_T \quad (63)$$

are $\mathbb{S}_n$-valued functions, with $G_1(T, \Theta) \succ 0$. Also,

$$\chi_1(\omega) := \chi'(\omega)(1 + \omega\chi'(\omega)), \quad \chi_2(\omega) := \omega\chi'(\omega) \quad (64)$$

are functions of a complex variable ω, which inherit from χ the analyticity in a neighbourhood of and real-valuedness on $\mathbb{R}_+$. Since $\chi_1(\omega) = \chi_2(\omega)(1 + \chi_2(\omega))/\omega$, then χ_1 or χ_2

is not constant. For any given (T, Θ), the map $\Sigma \mapsto \Omega_T$ in (58) is a bijection of $\mathbb{P}_n$, and hence, (61) is equivalent to

$$\sum_{k=1}^{2} \text{Tr}(G_k(T, \Theta)\chi_k(\Omega)) + \partial_T \rho_T = 0 \quad (65)$$

for all $T > 0$ and $\Theta, \Omega \succ 0$. Application of the separation-of-variables principle of Lemma 5 from Appendix B to (65) yields the existence of constants $\lambda, \tau \in \mathbb{R}$ such that the function χ, which generates χ_1, χ_2 in (64), satisfies the ODE

$$\chi'(\omega)(1 + \omega\chi'(\omega)) + \lambda\omega\chi'(\omega) + \tau = 0, \quad (66)$$

and G_1 and G_2 in (62) and (63) and ρ_T from (58) satisfy

$$(\partial_T \Psi_T)\Psi_T^{-1} + \Psi_T^{-1}\partial_T \Psi_T$$
$$- \Psi_T(A - D\Xi_T)\Psi_T^{-1} - \Psi_T^{-1}(A - D\Xi_T)^{\text{T}}\Psi_T = \lambda\Psi_T D\Psi_T, \quad (67)$$
$$\partial_T \rho_T = \tau\text{Tr}(D\Psi_T^2). \quad (68)$$

The PDEs (67) and (68) are solved in the lemma below and the result is then combined with the ODE (66).

Lemma 3: The following function is a solution of the modified HJE (57) in the class (58):

$$\widetilde{S}_T(\Sigma, \Theta) = \text{Tr}\chi(U_T(\Sigma)V_T(\Theta)) + \rho_T(\Theta), \quad (69)$$

where U_T is given by (48) and χ satisfies the ODE (66). Here, for any $\lambda \geqslant 0$, the map $V_T : \mathbb{P}_n \to \mathbb{P}_n$ is associated with (12) by

$$V_T(\Theta) := \Gamma_T^{-1/2}\mho_T(\Theta)^{-1}\Gamma_T^{-1/2}, \quad \mho_T(\Theta) := \lambda\Gamma_T^{-1} + \mho(\Theta), (70)$$

where $\mho(\Theta)$ is a $\mathbb{S}_n$-valued function of Θ only which satisfies

$$\mho(\Theta) \succcurlyeq -\lambda\Pi_*^{-1} \text{ for } \lambda > 0, \quad \mho(\Theta) \succ 0 \text{ for } \lambda = 0, \quad (71)$$

with Π_* given by (6). Also,

$$\rho_T(\Theta) = \rho(\Theta) - \begin{cases} (\tau/\lambda)\ln\det\mho_T(\Theta) & \text{for } \lambda > 0 \\ \tau\text{Tr}(\mho(\Theta)^{-1}\Gamma_T^{-1}) & \text{for } \lambda = 0 \end{cases}, \quad (72)$$

where $\rho(\Theta)$ is a $\mathbb{R}$-valued function of Θ only.

Proof: Since $\Psi^{-1}\dot{\Psi}\Psi^{-2} + \Psi^{-2}\dot{\Psi}\Psi^{-1} = \Psi^{-2}(\Psi^2)^{\cdot}\Psi^{-2} = -(\Psi^{-2})^{\cdot}$, left and right multiplication of both sides of (67) by Ψ_T^{-1} yields a differential Lyapunov equation

$$\partial_T(\Psi_T^{-2}) + (A - D\Xi_T)\Psi_T^{-2} + \Psi_T^{-2}(A - D\Xi_T)^{\text{T}} + \lambda D = 0 \quad (73)$$

with respect to Ψ_T^{-2}. Its solution is expressed in terms of the fundamental solution of the ODE

$$\partial_T \xi_T = -(A - D\Xi_T)\xi_T = -(A - De^{A^{\text{T}}T}\Gamma_T^{-1}e^{AT})\xi_T, \quad (74)$$

where Ξ_T is given by (56). The relation $\dot{\Gamma}_t = e^{At}De^{A^{\text{T}}t}$ implies that the general solution of (74) is $\xi_T = e^{-AT}\Gamma_T\zeta$, where $\zeta \in \mathbb{R}^n$ is an arbitrary constant vector. Hence, the solution of (73) can be found in the form

$$\Psi_T^{-2} = e^{-AT}\Gamma_T\mho_T\Gamma_T e^{-A^{\text{T}}T} \quad (75)$$

by the variation of constants method, where $\mho_T(\Theta)$ is a $\mathbb{S}_n$-valued function of T, Θ. Substitution of (75) into (73) yields

$$\partial_T\mho_T = -\lambda\Gamma_T^{-1}e^{AT}De^{A^{\text{T}}T}\Gamma_T^{-1} = \lambda\partial_T(\Gamma_T^{-1}), \quad (76)$$

whence $\mho_T$ in (70) is obtained by integration. Now, the right-hand side of (75) is positive definite for all $T > 0$ if and only if so is $\mho_T(\Theta)$. In view of (70), the condition $\mho_T(\Theta) \succ 0$ for any $T > 0$ is equivalent to $\lambda \geqslant 0$ and (71). Indeed, 1) the controllability Gramian in (12) satisfies $0 \prec \Gamma_T \prec \Pi_*$

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for all $T > 0$, and strictly $\prec$-monotonically approaches 0 and Π_* as T tends to 0 and $+\infty$, respectively; 2) $(\cdot)^{-1}$ is a decreasing operator on the set $\mathbb{P}_n$ with respect to the partial ordering $\prec$. Therefore,

$$\Psi_T(\Theta)^2 = \mathrm{e}^{A^\mathrm{T} T} \Gamma_T^{-1} \mho_T(\Theta)^{-1} \Gamma_T^{-1} \mathrm{e}^{AT}, \qquad (77)$$

which is obtained as the matrix inverse of (75). Substitution of (77) into (59) yields

$$\mathrm{Tr}\chi(\Omega_T) = \mathrm{Tr}\chi(\Sigma \mathrm{e}^{A^\mathrm{T} T} \Gamma_T^{-1} \mho_T(\Theta)^{-1} \Gamma_T^{-1} \mathrm{e}^{AT})$$
$$= \mathrm{Tr}\chi(\Gamma_T^{-1/2} \mathrm{e}^{AT} \Sigma \mathrm{e}^{A^\mathrm{T} T} \Gamma_T^{-1/2} \Gamma_T^{-1/2} \mho_T(\Theta)^{-1} \Gamma_T^{-1/2})$$
$$= \mathrm{Tr}\chi(U_T(\Sigma) V_T(\Theta)), \quad (78)$$

where we have used (48) and V_T from (70). By combining (77) with (68) and (76), it follows that

$$\partial_T \rho_T = \tau \mathrm{Tr}(D \mathrm{e}^{A^\mathrm{T} T} \Gamma_T^{-1} \mho_T(\Theta)^{-1} \Gamma_T^{-1} \mathrm{e}^{AT})$$
$$= \tau \mathrm{Tr}(\mho_T(\Theta)^{-1} \Gamma_T^{-1} \mathrm{e}^{AT} D \mathrm{e}^{A^\mathrm{T} T} \Gamma_T^{-1})$$
$$= - \begin{cases} (\tau/\lambda) \partial_T \ln \det \mho_T(\Theta) & \text{for } \lambda > 0 \\ \tau \partial_T \mathrm{Tr}(\mho(\Theta)^{-1} \Gamma_T^{-1}) & \text{for } \lambda = 0 \end{cases}. \quad (79)$$

Here, use is made of the property that, in the case $\lambda = 0$, the matrix $\mho_T(\Theta)$ from (70) does not depend on T. Integration of the PDE (79) yields (72). Assembling the latter and (78) into (58) leads to (69). $\square$

For any constant $s > 0$, the function $\widetilde{S}_T$, described by Lemma 3, is invariant under the scaling transformation

$$\chi(\omega) \mapsto \chi(s\omega), \quad \mho \mapsto s\mho, \quad \lambda \mapsto s\lambda, \quad \tau \mapsto s\tau, \quad (80)$$

and so are the ratio λ/τ and the ODE (66). We now combine Lemmas 2 and 3 to finalise the correction scheme (42)–(45).

Theorem 4: A nonnegative solution of the covariance HJE (40), satisfying the boundary condition (34), is given by

$$S_T(\Sigma, \Theta) := \ln \det(2U_T(\Sigma))$$
$$+ \mathrm{Tr}(U_T(\Sigma) + V_T(\Theta) + \chi(U_T(\Sigma) V_T(\Theta))). \quad (81)$$

Here,
$$\chi(\omega) = -\sqrt{1 + 4\omega} - \ln(\sqrt{1 + 4\omega} - 1), \qquad (82)$$

the map U_T is given by (48), and V_T in (70) takes the form
$$V_T(\Theta) = \Gamma_T^{-1/2} \Theta \Gamma_T^{-1/2}. \qquad (83)$$

Proof: By solving (66) as a quadratic equation with respect to $\chi'(\omega)$, it follows that

$$\chi'(\omega) = (-(\lambda\omega + 1) \pm \sqrt{(\lambda\omega + 1)^2 - 4\tau\omega})/(2\omega). \quad (84)$$

The function χ can now be obtained by integrating one of the two regular branches of (84) in ω. This integration is straightforward in the case $\lambda := 0$ which is shown below to yield a feasible solution in (81). In this case, τ must be negative to make $\chi(\omega)$ real for all $\omega \geqslant 0$ (the trivial situation $\lambda = \tau = 0$ is excluded from consideration). In view of (80), we set $\tau := -1$ without loss of generality. Integration of $\chi'(\omega) = -(1 + \sqrt{1 + 4\omega})/(2\omega)$ yields (82). This particular choice of the branch is motivated by the identity

$$\chi((v - 1)v) = 1 - 2v - \ln(2(v - 1)) \qquad (85)$$

for χ from (82), which is crucial in what follows to achieve the fulfillment of (34). To this end, by assembling the starting solution $\widehat{S}_T$ from Lemma 2 and the trace-analytic correcting

function $\widetilde{S}_T$ from Lemma 3 (with $\lambda := 0$, $\tau := -1$) into (42), it follows that

$$S_T(\Sigma, \Theta) = \ln \det U_T(\Sigma) + \mathrm{Tr} U_T(\Sigma)$$
$$+ \mathrm{Tr}\chi(U_T(\Sigma) V_T(\Theta)) + \rho(\Theta) + \mathrm{Tr} V_T(\Theta), \qquad (86)$$

where the map V_T is given by (70) with $\lambda = 0$, so that

$$V_T(\Theta) = \Gamma_T^{-1/2} \mho(\Theta)^{-1} \Gamma_T^{-1/2}. \qquad (87)$$

From (48), it follows that if the terminal state covariance matrix Θ is nominally reachable from the initial state covariance matrix Σ in time T, then $U_T(\Sigma) = \Gamma_T^{-1/2} \Theta \Gamma_T^{-1/2} - I_n$ for $\Theta = C_T(\Sigma)$. Hence, S_T in (86) satisfies (34) if and only if

$$S_T(C_T^{-1}(\Theta), \Theta) = \ln \det(\Theta \Gamma_T^{-1} - I_n) + \mathrm{Tr}(\Theta \Gamma_T^{-1} - I_n)$$
$$+ \mathrm{Tr}\chi((\Gamma_T^{-1/2} \Theta \Gamma_T^{-1/2} - I_n) V_T(\Theta)) + \rho(\Theta) + \mathrm{Tr} V_T(\Theta) = 0 \ (88)$$

for all $\Theta \succ \Gamma_T$. With χ given by (82), it is possible to find $\mho$, ρ so as to satisfy (88). Indeed, by setting $\mho(\Theta) := \Theta^{-1}$ and $\rho(\Theta) := n \ln 2$, the representation (87) becomes (83) and the boundary value of S_T from (88) takes the form

$$S_T(C_T^{-1}(\Theta), \Theta) = \ln \det(V_T(\Theta) - I_n) + \mathrm{Tr}(V_T(\Theta) - I_n)$$
$$+ \mathrm{Tr}\chi((V_T(\Theta) - I_n) V_T(\Theta)) + n \ln 2 + \mathrm{Tr} V_T(\Theta)$$
$$= \ln \det(2(V_T - I_n)) + \mathrm{Tr}(2V_T - I_n) + \mathrm{Tr}\chi((V_T - I_n) V_T).$$

The right-hand side of this equation vanishes for all nominally reachable $\Theta \succ \Gamma_T$ in view of (85). $\square$

The function S_T from Theorem 4 is a smooth solution of the covariance HJE (40). The minimum required supply in the state PDF transition problem (31) is obtained by combining Theorems 3 and 4:

$$J_T(\mathcal{N}(\alpha_0, \Pi_0), \mathcal{N}(\alpha_T, \Pi_T)) = (\|\alpha_T - M_T(\alpha_0)\|_{\Gamma_T^{-1}}^2$$
$$+ \mathrm{Tr}(U_T + V_T - \sqrt{I_n + 4U_T V_T})$$
$$- \ln \det((\sqrt{I_n + 4U_T V_T} - I_n)(2U_T)^{-1}))/2, \quad (89)$$

with $U_T = \Gamma_T^{-1/2} C_T(\Pi_0) \Gamma_T^{-1/2} - I_n$, $V_T = \Gamma_T^{-1/2} \Pi_T \Gamma_T^{-1/2}$. Here, M_T, C_T are the nominal state mean and covariance semigroups from (13), (14), and Γ_T is the finite-horizon controllability Gramian from (12). The structure of the right-hand side of (89) is identical to that in the discrete time case [17], except that the semigroups and the gramian are computed in accordance with the continuous time setting.

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APPENDIX

A. DIFFERENTIATION OF TRACE-ANALYTIC FUNCTIONS

Lemma 4: Let χ be a function of a complex variable, analytic in a neighbourhood of $\mathbb{R}_+$. Then the Frechet derivative of the function $\mathbb{P}_n \ni \Sigma \mapsto \mathrm{Tr}\chi(\Sigma) \in \mathbb{C}$ is

$$\partial_\Sigma \mathrm{Tr}\chi(\Sigma) = \chi'(\Sigma). \qquad (A1)$$

Proof: Applying an elementary polynomial $\chi(z) := z^k$ of degree $k \geqslant 1$ to a matrix $\Sigma \in \mathbb{S}_n$ yields the first variation $\delta\mathrm{Tr}(\Sigma^k) = \sum_{s=0}^{k-1} \mathrm{Tr}(\Sigma^s (\delta\Sigma)\Sigma^{k-1-s}) = k\mathrm{Tr}(\Sigma^{k-1}\delta\Sigma)$; cf. [15, p. 270]. By linearity, this proves (A1) for arbitrary polynomials χ. A standard passage to the limit extends (A1) to a power series $\chi(z) := \sum_{k=0}^{+\infty} c_k(z - z_0)^k$ whose disk of convergence contains the spectrum of Σ. $\square$

For the purposes of Section X, let Ψ be a smooth $\mathbb{P}_n$-valued function of the independent time variable T which generates a $\mathbb{P}_n$-valued function $\Omega := \Psi\Sigma\Psi$ of T and $\Sigma \in \mathbb{P}_n$. Then for any χ from Lemma 4,

$$(\mathrm{Tr}\chi(\Omega))^\bullet = \mathrm{Tr}((\dot\Psi\Psi^{-1} + \Psi^{-1}\dot\Psi)\chi'(\Omega)\Omega), \qquad (A2)$$

$$\partial_\Sigma \mathrm{Tr}\chi(\Omega) = \Psi\chi'(\Omega)\Psi, \qquad (A3)$$

where $(\dot{\ }) := \partial_T$. Indeed, (A2) is obtained by applying the chain rule to the composition of $\mathrm{Tr}\chi$ and Ω as a function of T for a fixed Σ and employing (A1): $(\mathrm{Tr}\chi(\Omega))^\bullet = \mathrm{Tr}(\chi'(\Omega)(\dot\Psi\Sigma\Psi + \Psi\Sigma\dot\Psi)) = \mathrm{Tr}((\dot\Psi\Psi^{-1} + \Psi^{-1}\dot\Psi)\chi'(\Omega)\Omega)$, where the commutativity of Ω and $\chi'(\Omega)$ is used. In a similar vein, the variation $\delta\mathrm{Tr}\chi(\Omega) = \mathrm{Tr}(\chi'(\Omega)\Psi(\delta\Sigma)\Psi) = \mathrm{Tr}(\Psi\chi'(\Omega)\Psi\delta\Sigma)$ with respect to Σ for a fixed T yields (A3).

B. SEPARATION OF VARIABLES FOR ANALYTIC FUNCTIONS OF MATRICES

Lemma 5: Let χ_1, χ_2 be functions of a complex variable, analytic in a neighbourhood of and real-valued on $\mathbb{R}_+$. Let G_1, G_2 be $\mathbb{S}_n$-valued functions of an independent variable T with $\mathrm{Tr}G_1(T_0) \neq 0$ for some value T_0 of T. Suppose that

$$\sum_{k=1}^{2} \mathrm{Tr}(G_k(T)\chi_k(\Omega)) = R(T) \quad \text{for all } \Omega \in \mathbb{S}_n^+ \text{ and } T, \quad (B1)$$

where $R(T)$ is a function of T only. Then χ_1, χ_2 are affinely dependent in their common analyticity domain:

$$\chi_1 + \lambda\chi_2 \equiv \tau, \qquad (B2)$$

where λ and τ are real constants. If, in addition to the previous assumptions, χ_1 or χ_2 is nonconstant, then the pair (λ, τ) is unique, and

$$G_2 \equiv \lambda G_1, \qquad R \equiv \tau\mathrm{Tr}G_1. \qquad (B3)$$

Proof: By considering (B1) for scalar matrices $\Omega = \omega I_n$, with $\omega \in \mathbb{R}_+$, it follows that

$$\sum_{k=1}^{2} \chi_k(\omega)\mathrm{Tr}G_k(T) = R(T) \quad \text{for all } \omega \geqslant 0 \text{ and } T. \quad (B4)$$

Dividing both sides of (B4) for $T = T_0$ by $\mathrm{Tr}G_1(T_0) \neq 0$ and introducing the ratios

$$\lambda := \mathrm{Tr}G_2(T_0)/\mathrm{Tr}G_1(T_0), \quad \tau := R(T_0)/\mathrm{Tr}G_1(T_0)$$

yields the affine dependence (B2) of the functions χ_1 and χ_2 on $\mathbb{R}_+$ which extends to their common analyticity domain by the identity theorem of complex analysis. Now, let χ_1 or χ_2 be nonconstant. Then (B2) determines the pair $(\lambda, \tau) \in \mathbb{R}^2$ uniquely. Therefore, the following implication holds for any $\mu := (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$:

$$\mu_1\chi_1 + \mu_2\chi_2 \equiv \mu_3 \implies \mu_2 = \lambda\mu_1, \ \mu_3 = \tau\mu_1. \qquad (B5)$$

For any $\omega \geqslant 0$ and $u \in \mathbb{R}^n$ with $|u| = 1$, consider a matrix $\Omega := \omega uu^T$. Its eigenvalues are ω (with the eigenvector u) and 0 (with the eigenspace $u^\perp$, the orthogonal complement of u in $\mathbb{R}^n$). The spectral decomposition of Ω yields $\chi(\omega uu^T) = \chi(\omega)uu^T + \chi(0)(I_n - uu^T)$, and (B1) takes the form

$$\sum_{k=1}^{2} u^T G_k(T)u\,\chi_k(\omega)$$

$$= R(T) - \sum_{k=1}^{2} \chi_k(0)(\mathrm{Tr}G_k(T) - u^T G_k(T)u), \quad \omega \geqslant 0. \qquad (B6)$$

Here, all ω-independent terms are moved to the right-hand side. By considering (B6) for fixed but otherwise arbitrary values of u and T and recalling (B5), it follows that

$$u^T G_2(T)u = \lambda u^T G_1(T)u, \qquad (B7)$$

$$R(T) - \sum_{k=1}^{2} \chi_k(0)(\mathrm{Tr}G_k(T) - u^T G_k(T)u) = \tau u^T G_1(T)u. \qquad (B8)$$

Since the unit vector u is arbitrary and the matrices $G_1(T)$ and $G_2(T)$ are both symmetric, then (B7) implies the first of the relations (B3). Combining the latter with (B2) gives

$$\sum_{k=1}^{2} \chi_k(0)(\mathrm{Tr}G_k(T) - u^T G_k(T)u) = (\chi_1(0) + \lambda\chi_2(0))$$

$$\times(\mathrm{Tr}G_1(T) - u^T G_1(T)u) = \tau(\mathrm{Tr}G_1(T) - u^T G_1(T)u). \qquad (B9)$$

Substitution of (B9) into (B8) yields

$$R(T) = \tau u^T G_1(T)u + \sum_{k=1}^{2} \chi_k(0)(\mathrm{Tr}G_k(T) - u^T G_k(T)u)$$

$$= \tau u^T G_1(T)u + \tau(\mathrm{Tr}G_1(T) - u^T G_1(T)u) = \tau\mathrm{Tr}G_1(T)$$

which, by the arbitrariness of T, establishes the second of the relations (B3). $\square$