Stable stochastic receding horizon control of linear systems with bounded control inputs

Peter Hokayem, Debasish Chatterjee, Federico Ramponi, Georgios Chaloulos, John Lygeros

***Abstract*— We address stability of receding horizon control for stochastic linear systems with additive noise and bounded control authority. We construct tractable and recursively feasible receding horizon control policies that ensure a mean-square bounded system in closed-loop if the noise has bounded forth-order moment, the unexcited system is stabilizable, the system matrix A is Lyapunov stable, and there is large enough control authority.**

I. Introduction

Receding horizon has become a natural way of dealing with the optimal control problem for linear systems involving input and state constraints [31], [29], [4], [15]. An inherent feature of receding-horizon methods is that the controllers may not be stabilizing in general. In the deterministic setting one starts with some initially feasible region and poses some terminal constraints which the state must satisfy at the end of the optimization horizon to ensure stability. In the stochastic setting involving possibly unbounded disturbances, however, no such terminal constraints can be posed. Moreover, imposing any bound on the control inputs with no restrictions on the system matrices may lead to the resulting system being unstable. At an intuitive level, this is quite natural since the control authority is restricted, while the disturbance can take large values, and can therefore drive the system unstable. In this article we aim to pose constraints under which receding horizon control policies can render the state of a given system *mean-square bounded*.

Receding horizon control of deterministic systems is a well-studied subject, see, e.g., [31], [4], [28], [15] and the references therein. In the stochastic setting, following early theoretical works on Markov decision processes [2], [21], [10], constructive methods with a strong applied flavor have received considerable attention. In [12], the authors first formulate the finite-horizon stochastic control problem as a deterministic one with bounded noise and then show that the underlying min-max problem is equivalent to a semidefinite program. In [36] a different problem is addressed in which the noise enters in a multiplicative manner, and hard constraints on the states and control inputs are relaxed to constraints resembling the integrated chance constraints of [20] or risk measures in mathematical finance. Another approach involving multiplicative noise and probabilistic input constraints has appeared in [16]. Similar relaxations of hard constraints to soft probabilistic ones have appeared in [33], [1] for additive noise inputs, as well as in [17] for both multiplicative and additive noise inputs. Alternative approaches employing randomized algorithms have been proposed in [39], [14], [3], [30]. Related lines of research can be found in [44], [45] dealing with constrained model predictive control (MPC) for stochastic linear systems motivated by industrial applications, in [41], [40] dealing with various tractable methods of controller synthesis, in [13] where optimality of a certain class of tractable feedback policies is established for deterministic scalar linear systems, and in [6] dealing with linear systems with multiplicative disturbances and using the scenario approach.

The deterministic version of the stabilization problem with bounded control inputs was solved completely in a sequence of articles [47], [43] culminating in [46]. It was demonstrated that global asymptotic stabilization of a discrete-time linear system $x_{t+1} = Ax_t + Bu_t$ with bounded feedback controls is possible if and only if the system matrix A has spectral radius at most 1, and the pair (A, B) is stabilizable with arbitrary controls. See also the recent work [25] dealing with input-to-state stability of bounded receding horizon controllers, and some earlier related work in [26]. In the context of stochastic linear systems $x_{t+1} = Ax_t + Bu_t + w_t$ with $(w_t)_{t=0,1,2,\cdots}$ being i.i.d noise vectors, it is possible to establish mean-square boundedness of the closed loop system under bounded receding horizon control policies by employing classical Foster-Lyapunov techniques [32] if the system matrix A is Schur stable [22], [18]. In this article we demonstrate that it is possible to strengthen this last result to the case of A being Lyapunov stable by considering the setup in the recent article by the authors [37]. It was shown in [37] that there exists a bounded control policy that ensures mean-square boundedness of the closed-loop system if the pair (A, B) is controllable, A is Lyapunov stable, and the process noise has bounded fourth moment. This is the track that we shall pursue. For related results see also [42].

The remainder of this article is organized as follows: We formulate the problem in Section II with all the underlying assumptions. In Section III we state the main result and prove it in Section IV. We conclude in Section V.

Notation

We denote by $\mathbb{N}$ the set of non-negative integers. We denote by $\|\cdot\|$ the standard Euclidean norm on $\mathbb{R}^n$ and by $\|\cdot\|_\infty$ the ℓ_∞ norm. For any two matrices A and B of

This research was partially supported by the Swiss National Science Foundation, grant 200021-122072, and FeedNetBack FP7-ICT-223866 (www.feednetback.eu).

ADDRESS: Automatic Control Laboratory, ETL I28, ETH Zürich, Physikstrasse 3, 8092 Zürich, Switzerland

URL: http://control.ee.ethz.ch

EMAILS: {hokayem, chatterjee, ramponif, chaloulos, lygeros}@control.ee.ethz.ch

ISBN 978-963-311-370-7

compatible dimensions, we denote by $\mathcal{R}_k(A,B)$ the k-th step reachability matrix $\mathcal{R}_k(A,B) := \begin{bmatrix} A^{k-1}B & \cdots & AB & B \end{bmatrix}$. For any matrix $M \in \mathbb{R}^{n\times m}$, we let $\sigma_1(M) \geq \cdots \geq \sigma_{\min\{n,m\}}(M)$ denote its singular values and $M^\dagger$ denote its Moore-Penrose pseudo-inverse. We also denote by $(M)_{i:j}$ the matrix comprising the i-th to j-th rows of M, where $j \geq i$. For a $n \times n$ positive semidefinite matrix M, we let $\|x\|_M := \sqrt{x^T M x}$ denote the weighted semi-norm of $x \in \mathbb{R}^n$. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a general probability space, we denote the conditional expectation given the sub-σ algebra $\mathfrak{F}'$ of $\mathfrak{F}$ as $\mathbb{E}^{\mathfrak{F}'}[.]$. In a Euclidean space we denote by $\mathcal{B}_r$ the closed Euclidean ball or radius r centered at the origin. For $r > 0$ let the saturation function $\mathrm{sat}_r : \mathbb{R}^d \longrightarrow \mathcal{B}_r$ be defined by $\mathrm{sat}_r(y) = y$ if $y \in \mathcal{B}_r$, and $\mathrm{sat}_r(y) = ry/\|y\|$ otherwise. Note that $\mathrm{sat}_r(\cdot)$ is *not* the component-wise saturation function.

II. Problem Setup

A. Dynamics and Cost

Consider the following discrete-time linear system with additive noise

$$x_{t+1} = Ax_t + Bu_t + w_t, \qquad x_0 = x, \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $(w_t)_{t\in\mathbb{N}}$ is a sequence of n-dimensional random vectors, and $(A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. We posit the following standing assumption:

Assumption 1:

1) The system matrices in (1) satisfy the following:
   a) (A,B) is stabilizable [7, Chapter 12];
   b) A is discrete-time Lyapunov stable [7, Chapter 12], i.e., the eigenvalues $\{\lambda_i(A) \mid i = 1,\ldots,d\}$ lie in the closed unit disc, and those eigenvalues $\lambda_j(A)$ with $|\lambda_j(A)| = 1$ have equal algebraic and geometric multiplicities;
2) The noise sequence $(w_t)_{t\in\mathbb{N}}$ in (1) is a collection of independent n-dimensional random vectors having bounded fourth moment, i.e., there exists $C_4 > 0$ such that $\mathbb{E}\left[\|w_t\|^4\right] \leq C_4 < \infty$ for all $t \in \mathbb{N}$.
3) The control inputs take values in the control constraint set
$$\mathcal{U} := \left\{\xi \in \mathbb{R}^m \,\middle|\, \|\xi\| \leq U_{\max}\right\}, \tag{2}$$
i.e., $u_t \in \mathcal{U}$ for all $t \in \mathbb{N}$. ♢

Select $C_1 > 0$ such that $\mathbb{E}[\|w_t\|] \leq C_1$ for all $t \in \mathbb{N}$; this is possible since Jensen's inequality leads to $\mathbb{E}[\|w_t\|] \leq \sqrt[4]{C_4}$.

Without any loss of generality, we assume that A is in real Jordan canonical form. Indeed, given a linear system described by system matrices $(\tilde{A}, \tilde{B})$, there exists a coordinate transformation in the state-space that brings the pair $(\tilde{A}, \tilde{B})$ to the pair (A,B), where A is in real Jordan form [24, p. 150]. In particular, choosing a suitable ordering of the Jordan blocks, we can ensure that the pair (A,B) has the form $\left(\begin{bmatrix} A_s & 0 \\ 0 & A_o \end{bmatrix}, \begin{bmatrix} B_s \\ B_o \end{bmatrix}\right)$, where $A_s \in \mathbb{R}^{n_s\times n_s}$ is Schur stable, and $A_o \in \mathbb{R}^{n_o\times n_o}$ has its eigenvalues on the unit circle. Due to the stability hypothesis Assumption 1-1b, A_o is therefore block-diagonal with elements on the diagonal being either ± 1 or 2×2 rotation matrices. As a consequence, A_o is orthogonal. Moreover, since (A,B) is stabilizable, the pair (A_o, B_o) must be reachable in a number of steps $\kappa \leq n_o$ that depends on the dimension of A_o and the structure of (A_o, B_o). Summing up, we can start by considering that system (1) has the form

$$\begin{bmatrix} x^s_{t+1} \\ x^o_{t+1} \end{bmatrix} = \begin{bmatrix} A_s x^s_t \\ A_o x^o_t \end{bmatrix} + \begin{bmatrix} B_s \\ B_o \end{bmatrix} u_t + \begin{bmatrix} w^s_t \\ w^o_t \end{bmatrix}, \tag{3}$$

where A_s is Schur stable and A_o is orthogonal. And, there exists a $\kappa \leq n_o$ such that the subsystem (A_o, B_o) is reachable in κ steps, i.e., $\mathrm{rank}\big(\mathcal{R}_\kappa(A_o, B_o)\big) = n_o$. This integer κ is fixed throughout the rest of the article, and we will consider it to be our *control horizon*.

We fix a second horizon, namely an *optimization (or prediction) horizon* $N \geq \kappa$. Given the state x_t at time t, let us consider the following objective function:

$$V_t := \mathbb{E}_{x_t}\left[\sum_{i=0}^{N-1}\left(\|x_{t+i}\|^2_{Q_i} + \|u_{t+i}\|^2_{R_i}\right) + \|x_{t+N}\|^2_{Q_N}\right], \tag{4}$$

where $Q_i \geq 0, R_i \geq 0, Q_N \geq 0$ are given symmetric matrices of appropriate dimensions. At times $t = 0, \kappa, 2\kappa, \cdots$, we are interested in minimizing (4) over the class of causal N-history-dependent strategies Π_N defined as:

$$\begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+N-1} \end{bmatrix} = \begin{bmatrix} \pi_t(x_t) \\ \pi_{t+1}(x_t, x_{t+1}) \\ \vdots \\ \pi_{t+N-1}(x_t, x_{t+1}, \cdots, x_{t+N-1}) \end{bmatrix}, \tag{5}$$

for some measurable functions $\{\pi_t, \cdots, \pi_{t+N-1}\} \in \Pi_N$, while satisfying the hard input constraints (2). The *receding horizon control* procedure with an optimization horizon N and a control horizon κ can be described as follows:

1) set $t = 0$;
2) measure the state x_t;
3) determine an admissible optimal control policy $\{\pi^*_t, \cdots, \pi^*_{t+N-1}\} \in \Pi_N$, that minimizes the N-stage cost function (4);
4) apply the first κ elements $\{\pi^*_t, \cdots, \pi^*_{t+\kappa-1}\}$ of the policy;
5) increase t to $t+\kappa$, and go back to step 2.

Therefore, the optimization problem to be solved at times $t = 0, \kappa, 2\kappa, \cdots$ is given by:

$$\min\left\{V_t \,\middle|\, (1),\ (2), \{\pi_t, \cdots, \pi_{t+N-1}\} \in \Pi_N\right\}. \tag{6}$$

The solution to Problem (6) is difficult to obtain in general. For instance, in order to obtain an optimal solution to Problem (6) over the class of causal state feedback policies, we need to solve the Dynamic Programming equations [9], [11]. This generally requires using gridding techniques, which suffer from the curse of dimensionality. Another approach is to restrict attention to a specific class of causal history-dependent policies. This will result in a suboptimal solution to our problem, but may yield a tractable optimization problem [8], [35]. It is this track that we pursue in the next section.

B. Policy Class

By the hypothesis that the state is fully observed without error, one may reconstruct the noise sequence from the sequence of observed states and inputs in Figure 1 by the formula

$$w_t = x_{t+1} - Ax_t - Bu_t, \qquad t \in \mathbb{N}. \tag{7}$$

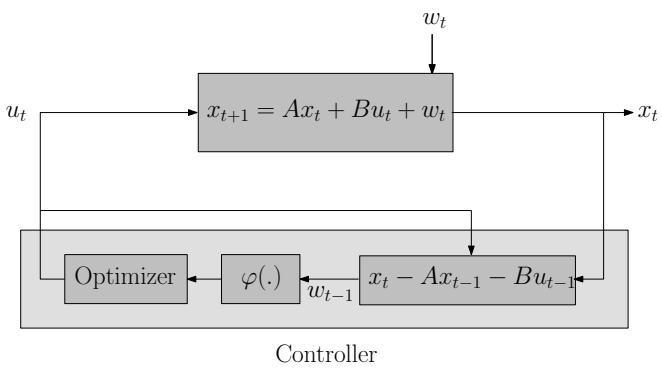

Fig. 1. Block diagram of the closed-loop system.

In light of this, we follow our earlier approach in [22], [18], [23] inspired by the works [38], [27], [5], [19] and use nonlinear N-history-dependent policies of the following form:

$$u_{t+l} = \eta_{t+l} + \sum_{i=0}^{l-1} \theta_{t+l,i}\varphi_i(w_{t+i}), \tag{8}$$

for $l = 0, 1, \ldots, N-1$ and a measured state x_t at time t, where $\varphi_i(w_i)$ is any vector-valued function of w_i such that $\|\varphi_i(w_i)\|_\infty \leq \varphi_{\max}$ for all i. In other words, we saturate the measurements that we obtain from the noise vectors before inserting them into our control vector. With this definition, the value of u at time $t+l$ depends on the values of w from time t up to time $t+l-1$. Therefore, a finite amount of memory is required to compute the value of the inputs u_t, for any $t \in \mathbb{N}$. We do not assume that the noise distribution is defined over a compact domain, which is a crucial assumption in robust MPC approaches [12], [19]. Moreover, the choice of element-wise saturation functions $\varphi_i(\cdot)$ is left open. As such, we can accommodate standard saturation, piecewise linear, and sigmoidal functions, to name a few.

Before we proceed to state the main result, it is helpful to have the following compact notation. The evolution of the system (1) over a single optimization or prediction horizon N, starting at t, can be described as follows:

$$X_t = \mathcal{A}x_t + \mathcal{B}U_t + \mathcal{D}W_t, \tag{9}$$

where $X_t = \begin{bmatrix} x_t \\ x_{t+1} \\ \vdots \\ x_{t+N} \end{bmatrix}$, $U_t = \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+N-1} \end{bmatrix}$, $W_t = \begin{bmatrix} w_t \\ w_{t+1} \\ \vdots \\ w_{t+N-1} \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} I \\ A \\ \vdots \\ A^N \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ B & \ddots & & \vdots \\ AB & B & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ A^{N-1}B & \cdots & AB & B \end{bmatrix}$, and $\mathcal{D} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & \ddots & & \vdots \\ A & I & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ A^{N-1} & \cdots & A & I \end{bmatrix}$. Similarly, the cost in (4) can be written as

$$V_t = \mathbb{E}_{x_t}\left[X_t^T \mathcal{Q} X_t + U_t^T \mathcal{R} U_t\right], \tag{10}$$

where $\mathcal{Q} = \text{diag}\{Q_0, Q_1, \cdots, Q_N\}$ and $\mathcal{R} = \text{diag}\{R_0, R_1, \cdots, R_{N-1}\}$. We can represent the control vectors over the optimization horizon as

$$U_t = \boldsymbol{\eta}_t + \boldsymbol{\Theta}_t \varphi(W_t), \tag{11}$$

where

$$\boldsymbol{\eta}_t := \begin{bmatrix} \eta_t \\ \eta_{t+1} \\ \vdots \\ \eta_{t+N-1} \end{bmatrix}, \quad \varphi(W_t) := \begin{bmatrix} \varphi_0(w_t) \\ \varphi_1(w_{t+1}) \\ \vdots \\ \varphi_{N-2}(w_{t+N-2}) \end{bmatrix},$$

and

$$\Theta_t := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \theta_{t+1,t} & 0 & \cdots & 0 \\ \theta_{t+2,t} & \theta_{t+2,t+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{t+N-1,t} & \theta_{t+N-1,t+1} & \cdots & \theta_{t+N-1,t+N-2} \end{bmatrix}. \tag{12}$$

Finally, the constraint (2) can be represented as

$$U_t \in \{\xi \in \mathbb{R}^{Nm} | \, \|\xi\|_\infty \leq U_{\max}\}. \tag{13}$$

III. Main Result

It is well known that receding horizon control is in general not stabilizing, even in the noise-free case and without any bounds on the control inputs [27, p.26]. This problem is due to the fact that the finite-horizon optimization problem does not necessarily inherit the infinite-horizon stability property, usually present in the traditional infinite horizon Linear Quadratic (LQ) control problem. This problem is circumvented in deterministic MPC by requiring that the state at the end of the optimization horizon enters some terminal positively invariant set [31]. In our setup, however, no such constraint can be enforced on the state as the process noise does not have compact support. Instead, we introduce an additional *stability constraint* which, if feasible, can render the state of the closed-loop system mean-square bounded. Guided by our approach in [37] we then show that this constraint is indeed recursively feasible.

We require that the following stability constraint be satisfied: for any given $\epsilon > 0$ and for every $t = 0, \kappa, 2\kappa, \cdots$, $U_t \in \mathcal{U}$ is chosen, such that the following "negative drift condition" holds:

$$\begin{aligned} &\left\| A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o) \begin{bmatrix} u_t \\ \vdots \\ u_{t+\kappa-1} \end{bmatrix} \right\| - \|x_t^o\| \\ &\leq -\left(\sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2}\right), \\ &\text{whenever } \|x_t^o\| \geq \sqrt{k}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \epsilon, \end{aligned} \tag{14}$$

where C_1 is as defined earlier. The constraint (14) pertains only to the orthogonal subsystem in (3). Taking the policy structure in (11) and the stability constraint (14) into account, the optimization problem to be solved at times $t = 0, \kappa, 2\kappa, \cdots$ becomes:

$$\min_{(\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t)} \left\{ V_t \,\middle|\, (9),\ (11),\ (13),\ \text{and } (14) \right\}. \tag{15}$$

Assumption 2: In addition to Assumption 1, we stipulate that the upper bound on the control authority $U_{\max} \geq U^*_{\max}$, where

$$U^*_{\max} := \sigma_{n_o}(\mathcal{R}_\kappa(A_o, B_o))^{-1} \left(\sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2} \right).$$

Theorem 3 (Main Result): Consider system (1) and suppose that Assumption 2 holds. Then:

(i) For every $t = 0, \kappa, 2\kappa, \cdots$, the optimization problem (15) is convex, feasible, and can be rewritten as the following tractable program with tightened constraints:

$$\begin{aligned} \min_{(\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t)} \ & \operatorname{tr}(\boldsymbol{\Theta}_t^T \mathcal{M} \boldsymbol{\Theta}_t \Lambda^{\varphi\varphi}) + 2\operatorname{tr}(\boldsymbol{\Theta}_t^T \mathcal{B}\mathcal{Q}\mathcal{D}\Lambda^{w\varphi}) \\ & + 2x_t^T \mathcal{A}^T \mathcal{Q}\mathcal{B}(\boldsymbol{\eta}_t + \boldsymbol{\Theta}_t \Lambda^\varphi) + \boldsymbol{\eta}_t^T \mathcal{M} \boldsymbol{\eta}_t \\ & + 2\boldsymbol{\eta}_t^T \mathcal{M} \boldsymbol{\Theta}_t \Lambda^\varphi \end{aligned} \tag{16}$$

subject to :

$$\boldsymbol{\Theta}_t \text{ having the structure in (12)}, \tag{17}$$

$$|(\boldsymbol{\eta}_t)_i| + \|(\boldsymbol{\Theta})_i\|_1 \varphi_{\max} \leq U_{\max} \ \forall i = 1, \cdots, Nm, \tag{18}$$

$$\begin{aligned} & \|A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\eta}_t)_{1:\kappa m}\| \\ & \quad + \sqrt{n_o}\, \|\mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\Theta}_t)_{1:\kappa m}\| \varphi_{\max} \\ & \quad - \|x_t^o\| \leq -\left(\sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2} \right) \\ & \quad \text{whenever } \|x_t^o\| \geq \sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \epsilon, \end{aligned} \tag{19}$$

where $\mathcal{M} = \mathcal{R} + \mathcal{B}^T \mathcal{Q}\mathcal{B}$, $\Lambda^\varphi = \mathbb{E}[\varphi(W_t)]$, $\Lambda^{w\varphi} = \mathbb{E}[W_t \varphi(W_t)^T]$, and $\Lambda^{\varphi\varphi} = \mathbb{E}[\varphi(W_t)\varphi(W_t)^T]$.

(ii) For any initial condition $x_0 \in \mathbb{R}^n$ successive application of the control laws given by the optimization problem in (i) for κ steps renders the closed-loop system mean-square bounded, i.e., there exists a constant $\gamma = \gamma(x_0, \epsilon, C_4, U_{\max})$ such that

$$\sup_{t \in \mathbb{N}} \mathbb{E}_{x_0}\left[\|x_t\|^2 \right] \leq \gamma. \tag{20}$$

A proof of Theorem 3 is provided in Section IV. Note that the constraint (18) is tightly equivalent to (13) (see [22]). The constraint (19) is a tightened representation of (14) for the control policy in (11), i.e., if (19) is satisfied then (14) is satisfied. The matrices Λ^φ, $\Lambda^{w\varphi}$, and $\Lambda^{\varphi\varphi}$ can be computed off-line as in [22], either in closed form or numerically, hence reducing the online computational burden.

IV. Proof of Theorem 3

Let us first recall the following fundamental result.

Proposition 4 ([34, Theorem 1]): Let $(\xi_t)_{t \in \mathbb{N}}$ be a sequence of nonnegative random variables on some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, and let $(\mathfrak{F}_t)_{t \in \mathbb{N}}$ be any filtration to which $(\xi_t)_{t \in \mathbb{N}}$ is adapted. Suppose that there exist constants $\epsilon > 0$, $J < \infty$ and $M < \infty$, such that $\xi_0 \leq J$, and for all t, we have the following two conditions satisfied:

$$\mathbb{E}^{\mathfrak{F}_t}\left[\xi_{t+1} - \xi_t \right] \leq -\frac{\epsilon}{2} \quad \text{on the event } \{\xi_t > J\}, \tag{21}$$

and

$$\mathbb{E}\left[|\xi_{t+1} - \xi_t|^4 \,|\, \xi_0, \ldots, \xi_t \right] \leq M. \tag{22}$$

Then there exists a constant $\gamma = \gamma(\epsilon, J, M) > 0$ such that

$$\sup_{t \in \mathbb{N}} \mathbb{E}\left[\xi_t^2 \right] \leq \gamma. \tag{23}$$

□

Proof of Theorem 3:

Proof of claim (i): The proof of convexity, the reformulation of the objective, and the constraints (17) and (18) can be found in [18], [22], and the constraint (19) is obviously convex as it is the sum of two norms. Therefore, it remains to show that the constraint (19) is both feasible and that it implies that the stability constraint (14) is satisfied.

Let us consider the feasibility of the constraint (19). Note that the matrix $\mathcal{R}_\kappa(A_o, B_o)$ has full rank equal to n_o. At any time $t = 0, \kappa, 2\kappa, \cdots$, consider the sequence of control input vectors

$$\tilde{u}_{t,t+\kappa-1} := \begin{bmatrix} \tilde{u}_t \\ \tilde{u}_{t+1} \\ \vdots \\ \tilde{u}_{t+\kappa-1} \end{bmatrix} := -\mathcal{R}_\kappa(A_o, B_o)^\dagger \operatorname{sat}_r(A_o^\kappa x_t^o),$$

for $r = \left(\sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2} \right)$. Substituting the input above into the left-hand side of (14), we obtain

$$\begin{aligned} & \left\| A_o^\kappa x_t^o - \mathcal{R}_\kappa(A_o, B_o)\mathcal{R}_\kappa(A_o, B_o)^\dagger \operatorname{sat}_r\left(A_o^\kappa x_t^o\right) \right\| - \|x_t^o\| \\ & = \left\| A_o^\kappa x_t^o - \operatorname{sat}_r\left(A_o^\kappa x_t^o\right) \right\| - \|A_o^\kappa x_t^o\| \\ & = -r = -\left(\sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2} \right), \end{aligned}$$

whenever $\|x_t^o\| \geq (\sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \epsilon)$. Moreover,

$$\begin{aligned} & \|\tilde{u}_{t,t+\kappa-1}\|_\infty \leq \|\tilde{u}_{t,t+\kappa-1}\|_2 \\ & \leq \left\| \mathcal{R}_\kappa(A_o, B_o)^\dagger \right\|_2 \|\operatorname{sat}_r(A_o^\kappa x_t^o)\|_2 \\ & \leq \sigma_{n_o}(\mathcal{R}_\kappa(A_o, B_o))^{-1} \left(\sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2} \right) \\ & = U^*_{\max} \leq U_{\max}. \end{aligned}$$

Therefore, we have shown that there exists an sequence of input vectors $\tilde{u}_{t,t+\kappa-1}$ that is feasible with respect to the input constraint set (2). Moreover, this policy is encompassed by the general policy structure (11), as we can set the design parameters to $(\tilde{\boldsymbol{\eta}}_t)_{1:\kappa m} = \tilde{u}_{t,t+\kappa-1}$ and $(\tilde{\boldsymbol{\Theta}}_t)_{1:\kappa m} = 0$.

Now, the sequence of control inputs can be written in terms of the decision variables as

$$u_{t,t+\kappa-1} := \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+\kappa-1} \end{bmatrix} = (\boldsymbol{\eta}_t)_{1:\kappa m} + (\boldsymbol{\Theta}_t)_{1:\kappa m}\varphi(W_t). \tag{24}$$

Combining (24) with the stability constraint (14), we obtain

$$\begin{aligned}
&\|A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o)u_{t,t+\kappa-1}\| \\
&\quad = \|A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\eta}_t)_{1:\kappa m} \\
&\qquad + \mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\Theta}_t)_{1:\kappa m}\varphi(W_t)\| \\
&\quad \le \|A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\eta}_t)_{1:\kappa m}\| \\
&\qquad + \|\mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\Theta}_t)_{1:\kappa m}\varphi(W_t)\| \\
&\quad \le \|A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\eta}_t)_{1:\kappa m}\| \\
&\qquad + \sqrt{n_o}\,\|\mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\Theta}_t)_{1:\kappa m}\varphi(W_t)\|_\infty \\
&\quad \le \|A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\eta}_t)_{1:\kappa m}\| \\
&\qquad + \sqrt{n_o}\,\|\mathcal{R}_\kappa(A_o, B_o)(\boldsymbol{\Theta}_t)_{1:\kappa m}\|_\infty\,\varphi_{\max}.
\end{aligned}$$

Accordingly, if the constraint (19) is satisfied, then the stability constraint (14) is satisfied as well. This completes the proof of claim (i) of Theorem 3.

Proof of claim (ii): In order to show mean-square boundedness, we start by dividing the problem into two parts:

$$\begin{aligned}
\sup_{t\in\mathbb{N}} \mathbb{E}_{x_0}\left[\|x_t\|^2\right] &= \sup_{t\in\mathbb{N}}\left(\mathbb{E}_{x_0}\left[\|x_t^s\|^2\right] + \mathbb{E}_{x_0}\left[\|x_t^o\|^2\right]\right) \\
&\le \sup_{t\in\mathbb{N}} \mathbb{E}_{x_0}\left[\|x_t^s\|^2\right] + \sup_{t\in\mathbb{N}} \mathbb{E}_{x_0}\left[\|x_t^o\|^2\right].
\end{aligned}$$

It has been shown in [18] that for the Schur stable subsystem, under bounded control inputs and noise sequence with bounded second moment, there exists a constant $\gamma_s = \gamma_s(x_0, C_4, U_{\max}) > 0$ such that

$$\sup_{t\in\mathbb{N}} \mathbb{E}_{x_0}\left[\|x_t^s\|^2\right] \le \gamma_s. \tag{25}$$

It remains to show that there exists a constant $\gamma_o = \gamma_o(x_0, \epsilon, C_4, U_{\max}) > 0$ such that

$$\sup_{t\in\mathbb{N}} \mathbb{E}_{x_0}\left[\|x_t^o\|^2\right] \le \gamma_o. \tag{26}$$

We shall rely on the result in Proposition (4) and show that both conditions (21) and (22) are satisfied from which it follows that the orthogonal subsystem has bounded variance.

Consider the sub-sampled process $(x_t^o)_{t=0,\kappa,2\kappa,\cdots}$, given by

$$\begin{aligned}
x_{t+\kappa}^o = A_o^k x_t^o + \mathcal{R}_k(A_o, B_o)u_{t,t+\kappa-1} + \\
\mathcal{R}_\kappa(A_o, I_{n_o})w_{t,t+\kappa-1}^o,
\end{aligned} \tag{27}$$

where $u_{t,t+\kappa-1} = \begin{bmatrix} u_t \\ \vdots \\ u_{t+\kappa-1} \end{bmatrix}$ and $w_{t,t+\kappa-1}^o = \begin{bmatrix} w_t^o \\ \vdots \\ w_{t+\kappa-1}^o \end{bmatrix}$.

Let $\mathfrak{F}_t := \sigma(x_0, \cdots, x_t)$ and observe that

$$\begin{aligned}
&\mathbb{E}^{\mathfrak{F}_t}\left[\left\|x_{t+\kappa}^o\right\| - \|x_t^o\|\right] \\
&= \mathbb{E}^{\mathfrak{F}_t}\Big[\Big\|A_o^k x_t^o + \mathcal{R}_\kappa(A_o, B_o)u_{t,t+\kappa-1} \\
&\qquad + \mathcal{R}_\kappa(A_o, I_{n_o})w_{t,t+\kappa-1}^o\Big\| - \|x_t^o\|\Big] \\
&\le \mathbb{E}^{\mathfrak{F}_t}\left[\left\|A_o^k x_t^o + \mathcal{R}_\kappa(A_o, B_o)u_{t,t+\kappa-1}\right\| - \|x_t^o\|\right] \\
&\quad + \mathbb{E}\left[\left\|\mathcal{R}_\kappa(A_o, I_{n_o})w_{t,t+\kappa-1}^o\right\|\right].
\end{aligned}$$

Now using the stability condition (14), we can establish the upper bound:

$$\begin{aligned}
\mathbb{E}^{\mathfrak{F}_t}\left[\left\|x_{t+\kappa}^o\right\| - \|x_t^o\|\right] &\le -\left(\sqrt{k}\sigma_1(\mathcal{R}_k(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2}\right) \\
&\quad + \mathbb{E}\left[\left\|\mathcal{R}_k(A_o, I_{n_o})w_{t,t+\kappa-1}^o\right\|\right] \\
&\le -\frac{\epsilon}{2},
\end{aligned}$$

for all $x_t^o \in \mathbb{R}^{n_o}$ such that $\|x_t^o\| > J := \sqrt{\kappa}\sigma_1(\mathcal{R}_\kappa(A_o, I_{n_o})C_1 + \varepsilon$, and condition (21) holds.

Also observe that

$$\begin{aligned}
&\mathbb{E}\left[\left(\left\|x_{t+\kappa}^o\right\| - \|x_t^o\|\right)^4 \middle| \{\|x_i^o\|\}_{i=0}^t\right] \\
&= \mathbb{E}\Big[\Big(\|A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o)u_{t,t+\kappa-1} \\
&\qquad + \mathcal{R}_\kappa(A_o, I_{n_o})w_{t,t+\kappa-1}^o\| - \|A_o^\kappa x_t^o\|\Big)^4 \Big| \{\|x_i^o\|\}_{i=0}^t\Big] \\
&\le \mathbb{E}\Big[\Big(\|A_o^\kappa x_t^o + \mathcal{R}_\kappa(A_o, B_o)u_{t,t+\kappa-1}\| - \|A_o^\kappa x_t^o\| \\
&\qquad + \left\|\mathcal{R}_\kappa(A_o, I_{n_o})w_{t,t+\kappa-1}^o\right\|\Big)^4 \Big| \{\|x_i^o\|\}_{i=0}^t\Big] \\
&\le \mathbb{E}\Big[\Big(\|\mathcal{R}_\kappa(A_o, B_o)u_{t,t+\kappa-1}\| \\
&\qquad + \left\|\mathcal{R}_\kappa(A_o, I_{n_o})w_{t,t+\kappa-1}^o\right\|\Big)^4 \Big| \{\|x_i^o\|\}_{i=0}^t\Big].
\end{aligned} \tag{28}$$

Since the control inputs are bounded by design and the forth moment of w_t is bounded by Assumption 1-2, expanding the last inequality in (28) and applying Jensen's inequality shows that there exists a constant $M = M(C_4, U_{\max}) > 0$ such that $\mathbb{E}\left[\left(\left\|x_{t+\kappa}^o\right\| - \|x_t^o\|\right)^4 \middle| \|x_0^o\|, \ldots, \|x_t^o\|\right] \le M$. As such both hypotheses of Proposition (4) are satisfied with $\xi_t := \|x_t^o\|$, and hence the sub-sampled process $(x_t^o)_{t=0,\kappa,2\kappa,\cdots}$ is mean-square bounded, i.e., there exists a constant $\bar{\gamma}_o = \bar{\gamma}_o(x_0, \epsilon, C_4, U_{\max}) > 0$ such that

$$\sup_{t=0,\kappa,2\kappa,\cdots} \mathbb{E}_{x_0}\left[\|x_t^o\|^2\right] \le \bar{\gamma}_o. \tag{29}$$

It follows from (29) via standard calculations involving triangle inequalities and the system dynamics that there exists another constant $\gamma_o = \gamma_o(\bar{\gamma}_o) > \bar{\gamma}_o$, such that (26) holds. Finally, setting $\gamma = \gamma_s + \gamma_o$ completes the proof of claim (ii) and hence the proof of Theorem (3). ■

V. Conclusions

We presented a method for stochastic receding horizon control of discrete-time linear systems with process noise and bounded control inputs. We showed that the optimization problem to be solved periodically is recursively feasible and convex under a suitable choice of control policies. Moreover, we showed how the optimization problem can be augmented with a certain stability constraint in order to render the state of the closed-loop system mean-square bounded. Future work will focus on extending this setup to the case of output feedback.

References

[1] M. Agarwal, E. Cinquemani, D. Chatterjee, and J. Lygeros. On convexity of stochastic optimization problems with constraints. In *European Control Conference*, pages 2827–2832, 2009.

[2] J. M. Alden and R. L. Smith. Rolling horizon procedures in nonhomogeneous Markov decision processes. *Operations Research*, 40(suppl. 2):S183–S194, 1992.

[3] I. Batina, A. A. Stoorvogel, and S. Weiland. Stochastic disturbance rejection in model predictive control by randomized algorithms. In *Proceedings of the American Control Conference*, volume 2, pages 732–737, 2001.

[4] A. Bemporad and M. Morari. Robust model predictive control: a survey. *Robustness in Identification and Control*, 245:207–226, 1999.

[5] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004.

[6] D. Bernardini and A. Bemporad. Scenario-based model predictive control of stochastic constrained linear systems. In *joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 6333–6338, Shanghai, P.R. China, Dec. 2009.

[7] D. S. Bernstein. *Matrix Mathematics*. Princeton University Press, 2 edition, 2009.

[8] D. Bertsekas and J. Tsitsiklis. *Neuro-Dynamic Programming*. Athena Scientific, 1996.

[9] D. P. Bertsekas. *Dynamic Programming and Optimal Control*, volume 1. Athena Scientific, 2 edition, 2000.

[10] D. P. Bertsekas. Dynamic programming and suboptimal control: a survey from ADP to MPC. *European Journal of Control*, 11(4-5):310–334, 2005.

[11] D. P. Bertsekas. *Dynamic Programming and Optimal Control*, volume 2. Athena Scientific, 3 edition, 2007.

[12] D. Bertsimas and D. B. Brown. Constrained stochastic LQC: a tractable approach. *IEEE Transactions on Automatic Control*, 52(10):1826–1841, 2007.

[13] D. Bertsimas, D. A. Iancu, and P. A. Parrilo. Optimality of affine policies in multi-stage robust optimization. http://arxiv.org/abs/0904.3986, 2009.

[14] L. Blackmore and B. C. Williams. Optimal, robust predictive control of nonlinear systems under probabilistic uncertainty using particles. In *Proceedings of the American Control Conference*, pages 1759–1761, 2007.

[15] E. F. Camacho and C. Bordons. *Model Predictive Control*. Springer-Verlag, London, 2 edition, 2004.

[16] M. Cannon, B. Kouvaritakis, and X. Wu. Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints. *Automatica*, 45(1):167–172, Jan. 2009.

[17] M. Cannon, B. Kouvaritakis, and X. Wu. Probabilistic constrained mpc for multiplicative and additive stochastic uncertainty. *IEEE Transactions on Automatic Control*, 54(7):1626–1632, July 2009.

[18] D. Chatterjee, P. Hokayem, and J. Lygeros. Stochastic receding horizon control with bounded control inputs—a vector-space approach. *IEEE Transactions on Automatic Control*, 2009. Under review. http://arxiv.org/abs/0903.5444.

[19] P. J. Goulart, E. C. Kerrigan, and J. M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42(4):523–533, 2006.

[20] W. K. K. Haneveld. On integrated chance constraints. In *Stochastic programming (Gargnano)*, volume 76 of *Lecture Notes in Control and Inform. Sci.*, pages 194–209. Springer, Berlin, 1983.

[21] O. Hernández-Lerma and J. B. Lasserre. Error bounds for rolling horizon policies in discrete-time Markov control processes. *IEEE Transactions on Automatic Control*, 35(10):1118–1124, 1990.

[22] P. Hokayem, D. Chatterjee, and J. Lygeros. On stochastic receding horizon control with bounded control inputs. In *joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 6359–6364, Shanghai, P.R. China, Dec. 2009. http://arxiv.org/abs/0902.3944.

[23] P. Hokayem, E. Cinquemani, D. Chatterjee, and J. Lygeros. Stochastic MPC with output feedback and bounded control inputs. In *American Control Conference*, Baltimore, MD, USA, Jun. 2010. Under review.

[24] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990.

[25] J.-S. Kim, T.-W. Yoon, A. Jadbabaie, and C. D. Persis. Input-to-state stable finite horizon MPC for neutrally stable linear discrete-time systems with input constraints. *Systems & Control Letters*, 55(4):293–303, 2006.

[26] Z. Lin, A. Saberi, and A. A. Stoorvogel. Semi-global stabilization of linear discrete-time systems subject to input saturation via linear feedback—an ARE-based approach. *IEEE Transactions on Automatic Control*, 41(8):1203–1207, 1996.

[27] J. Löfberg. *Minimax Approaches to Robust Model Predictive Control*. PhD thesis, Linköpings Universitet, 2003.

[28] J. M. Maciejowski. *Predictive Control with Constraints*. Prentice Hall, 2001.

[29] J. M. Maciejowski. *Predictive Control with Constraints*. Prentice Hall, 2002.

[30] J. M. Maciejowski, A. Lecchini, and J. Lygeros. NMPC for complex stochastic systems using Markov Chain Monte Carlo. In *International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, volume 358/2007 of *Lecture Notes in Control and Information Sciences*, pages 269–281, Stuttgart, Germany, 2005. Springer.

[31] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: stability and optimality. *Automatica*, 36(6):789–814, 2000.

[32] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability*. Springer-Verlag, London, 1993.

[33] F. Oldewurtel, C. Jones, and M. Morari. A tractable approximation of chance constrained stochastic MPC based on affine disturbance feedback. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 4731–4736, 2008.

[34] R. Pemantle and J. S. Rosenthal. Moment conditions for a sequence with negative drift to be uniformly bounded in L^r. *Stochastic Processes and their Applications*, 82(1):143–155, 1999.

[35] W. B. Powell. *Approximate Dynamic Programming*. Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2007.

[36] J. A. Primbs and C. H. Sung. Stochastic receding horizon control of constrained linear systems with state and control multiplicative noise. *IEEE Transactions on Automatic Control*, 54(2):221–230, 2009.

[37] F. Ramponi, D. Chatterjee, A. Milias-Argeitis, P. Hokayem, and J. Lygeros. Attaining mean square boundedness of a marginally stable noisy linear system with a bounded control input. *IEEE Transactions on Automatic Control*, 2009. Under review. http://arxiv.org/abs/0907.1436.

[38] R. T. Rockafellar and R. J.-B. Wets. The optimal recourse problem in discrete time: L^1-multipliers for inequality constraints. *SIAM Journal on Control and Optimization*, 16(1):16–36, 1978.

[39] E. Siva, P. Goulart, J. M. Maciejowski, and N. Kantas. Stability of model predictive control using Markov chain Monte Carlo optimisation. In *Proceedings of the European Control Conference*. European Union Control Organisation, 2009.

[40] J. Skaf and S. Boyd. Design of affine controllers via convex optimization. http://www.stanford.edu/~boyd/papers/affine_contr.html, 2009. To appear in IEEE Transactions on Automatic Control.

[41] J. Skaf and S. Boyd. Nonlinear Q-design for convex stochastic control. *IEEE Transactions on Automatic Control*, 54(10):2426–2430, 2009.

[42] A. A. Stoorvogel, A. Saberi, and S. Weiland. On external semi-global stochastic stabilization of linear systems with input saturation. In *Proceedings of the American Control Conference*, pages 5845–5850, 2007.

[43] H. J. Sussmann, E. D. Sontag, and Y. Yang. A general result on the stabilization of linear systems using bounded controls. *IEEE Transactions on Automatic Control*, 39(12):2411–2425, 1994.

[44] D. H. van Hessem and O. H. Bosgra. A full solution to the constrained stochastic closed-loop MPC problem via state and innovations feedback and its receding horizon implementation. In *Proceedings of the 42nd IEEE Conference on Decision and Control*, volume 1, pages 929–934, 2003.

[45] D. H. van Hessem and O. H. Bosgra. Stochastic closed-loop model predictive control of continuous nonlinear chemical processes. *Journal of Process Control*, 16(3):225–241, 2006.

[46] Y. Yang, E. D. Sontag, and H. J. Sussmann. Global stabilization of linear discrete-time systems with bounded feedback. *Systems & Control Letters*, 30(5):273–281, 1997.

[47] Y. Yang, H. J. Sussmann, and E. D. Sontag. Stabilization of linear systems with bounded controls. In M. Fliess, editor, *Proceedings of the Nonlinear Control Systems Design Symposium*, pages 15–20. IFAC Publications, 1992.